Complex Ginzburg-Landau Equations as Perturbations of Nonlinear Schrödinger Equations: Traveling Wave Persistence

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Abstract

The criteria for persistence of spatially periodic traveling waves of generalized nonlinear Schrödinger equations under generalized Ginzburg-Landau perturbations is studied in detail. We first develope a thorough classification of the periodic traveling wave solutions of the unperturbed nonlinear Schrödinger equations, to allow detailed refinements of the persistence criteria for the various types of traveling waves. We then consider persistence criteria. In a prior paper, we derived necessary conditions for traveling wave persistence directly from the partial differential equations via an averaging technique. Here we show that these conditions are identical to the Melnikov conditions derived from the ordinary differential equations that govern the traveling wave profiles, and we extend our prior results by deriving sufficient criteria for traveling wave persistence. These criteria are then specialized to the different types of traveling waves.

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1 Introduction

This paper is the second in a series of three papers (see also [1,2]) and also some other work by some of us [4–7] in a program concerned with the long-time dynamics of a class of infinite dimensional Hamiltonian systems when the symmetries of these systems are broken by a dissipative perturbation.

Specifically, in this series we study the peristence of solutions of the generalized nonlinear Schrödinger (GNLS) equation

$$\partial_t A = i\partial_{xx} A - ih'(|A|^2)A\,, \tag{1.1}$$

when the GNLS equation is perturbed to a generalized complex Ginzburg-Landau (GCGL) equation

$$\partial_t A = (i+\epsilon)\partial_{xx} A - ih'(|A|^2)A - \epsilon\, g'(|A|^2)A\,. \tag{1.2}$$

Here $\epsilon > 0$, while $h = h(\xi)$ and $g = g(\xi)$ are real analytic functions over $[0, \infty)$. Necessary conditions for the persistence of general spatially periodic, temporally quasiperidic and homoclinic solutions were derived in [1].

In this paper we examine in much greater depth the persistence of traveling wave solutions of (1.1). By a traveling wave, we mean a solution $A(x,t)$ of (1.1) of the form

$$A = Q(x-ct)e^{-i\alpha t + iS(x-ct)}\,, \tag{1.3}$$

where α and c real constants and S and Q are periodic real-valued functions with some period T. In [1] it was shown directly from the GNLS and GCGL equations that the following conditions are necessary for a GNLS traveling wave to persist as GCGL traveling wave of the same period,

$$M^{\mathcal{M}} \equiv \int_0^T \Big((\partial_z Q)^2 + Q^2(\partial_z S)^2 + g'(Q^2)Q^2 \Big)\, dz = 0\,, \tag{1.4}$$

$$M^{\mathcal{J}} \equiv \int_0^T \Big(Q^2(\partial_z S)^3 - 2Q(\partial_{zz} Q)\partial_z S + (\partial_z Q)^2 \partial_z S - g'(Q^2)Q^2\partial_z S \Big)\, dz = 0\,. \tag{1.5}$$

Here the superscripts $\mathcal{M}$ and $\mathcal{J}$ stand for the mass and momentum functionals

$$\mathcal{M} \equiv \int_0^T |A|^2 \, dx \,, \qquad \mathcal{J} \equiv \frac{1}{2i} \int_0^T (A^* \partial_x A - A \partial_x A^*) \, dx \,, \tag{1.6}$$

with which the conditions were derived. The quantities $\mathcal{M}$ and $\mathcal{J}$ are both conserved quantities of the GNLS equation (1.1). As discussed in [1], satisfaction of conditions (1.4) and (1.5) actually implies the satisfaction of an infinite number of necessary conditions derived from an infinite number of conserved quantities (underlying symmetries) of the GNLS equation.

The main task of the present paper is to show that these necessary conditions are identical to those obtained from the ordinary differential equations governing the profiles Q and S, and also to obtain *sufficient* conditions for traveling wave persistence. Another task of the paper is to classify the GNLS traveling waves, which is essential to the refinement of the selection criteria. These tasks form the basis for a case study in the next paper [2] of this series.

The rest of this paper is organized as follows. In section 2 the traveling wave solutions of the GNLS equation are classified into several different types of waves, and the notion of families of these solutions is precisely defined. Then in sections 3 and 4 the necessary and sufficient conditions for the persistence of the two basic types of these waves, which we call uncentered and linear phase waves, are treated seperately.

2 Families of GNLS Traveling Wave Solutions

In this section we classify the spatially periodic traveling wave solutions of the GNLS equation (1.1), define the notion of a family of traveling waves, and characterize their parameterization.

We begin with a rough parameter count. Writing

$$A(x,t) = e^{-i\alpha t} B(x - ct) \,, \tag{2.1}$$

where B is a periodic complex-valued function of $z = x - ct$ of period T, and substituting this into the GNLS equation (1.1), it follows that B solves the profile equation

$$i\partial_{zz} B + i\alpha B + c\partial_z B - ih'(|B|^2) B = 0 \,. \tag{2.2}$$

The question of existence then reduces to the question of existence of periodic solutions to equation (2.2). Equation (2.2) is a second order complex-valued ODE and therefore has four real constants of integration. Two of these constants of integration are trivial; they correspond to the translational invariance in z and in the phase. Counting the two parameters α and c, traveling waves will therefore be parameterized by four nontrivial parameters, two of which will be quantized by the periodicity condition. Thus we expect at the outset that the periodic traveling waves will be parameterized by three continuous parameters (one of which is the period), and two discrete parameters.

We now write B in polar form $B = Qe^{iS}$ (as in (1.3)), where Q and S are real.

We will assume throughout the rest of the paper that the trivial integration constants have been chosen such that

$$S(0) = 0 \,, \qquad Q(0) = Q_{\max} \,. \tag{2.3}$$

The real and imaginary parts of the profile equation (2.2) yield the system

$$\partial_{zz}Q - Q(\partial_z S)^2 + cQ\partial_z S + \alpha Q - h'(Q^2)Q = 0 \,, \tag{2.4}$$

$$\partial_z\left(Q^2\left(\partial_z S - \frac{c}{2}\right)\right) = 0 \,. \tag{2.5}$$

Integrating (2.5) gives

$$Q^2\left(\partial_z S - \frac{c}{2}\right) = \mu \,, \tag{2.6}$$

where μ is a constant of integration. Under the proviso that $Q > 0$ if $\mu \neq 0$, which is justified below, this relation expresses $\partial_z S$ in terms of Q as

$$\partial_z S = \frac{c}{2} + \frac{\mu}{Q^2} \,. \tag{2.7}$$

Substituting this expression into (2.4), the equation for Q reduces to

$$\partial_{zz}Q - \frac{\mu^2}{Q^3} + \gamma Q - h'(Q^2)Q = 0 \,, \tag{2.8}$$

where

$$\gamma \equiv \frac{c^2}{4} + \alpha \,. \tag{2.9}$$

This is a second order real-valued ODE. One of its two constants of integration corresponds again to the translational invariance.

Defining $R \equiv \partial_z Q$, equation (2.8) can be written as a Hamiltonian system in the form

$$\partial_z Q = \frac{\partial H}{\partial R}, \tag{2.10}$$

$$\partial_z R = -\frac{\partial H}{\partial Q}, \tag{2.11}$$

with Hamiltonian

$$H = \frac{1}{2}\left(R^2 + V(Q)\right), \tag{2.12}$$

and where the potential V is given by

$$V(Q; \mu, \gamma) = \frac{\mu^2}{Q^2} + \gamma Q^2 - h(Q^2). \tag{2.13}$$

We now describe the possible types of solutions of the Hamiltonian system (2.10–2.13) occurring at different values of the parameters and the effective energy

$$\beta \equiv 2H. \tag{2.14}$$

The potential V is always an even function of Q. Therefore a periodic orbit is either of one sign or symmetric with respect to the origin. Without loss of generality, for orbits of one sign we will consider only the positive sign in expressions. Note that the case of symmetric orbits can only happen when $\mu = 0$ because V diverges as $Q \to 0$ when $\mu \neq 0$. These two cases are illustrated in Figure 1, along with various solution types that we now describe.

We introduce some terminology here to keep track of the different types of traveling waves, which can have different persistence properties under perturbation. If $\mu = 0$, the normalized phase S is linear in z, and by (2.7) and (2.3) is given by

$$S(z) = \frac{c}{2} z. \tag{2.15}$$

Thus, we call all solutions with $\mu = 0$ 'linear phase' waves. We call solutions with $\mu \neq 0$ 'nonlinear phase' waves.

If the function Q is of one sign, and therefore asymmetric with respect to the origin, then we also call the wave 'uncentered'. Thus we can have uncentered linear phase waves (labeled ULPW in Figure 1a) and uncentered nonlinear

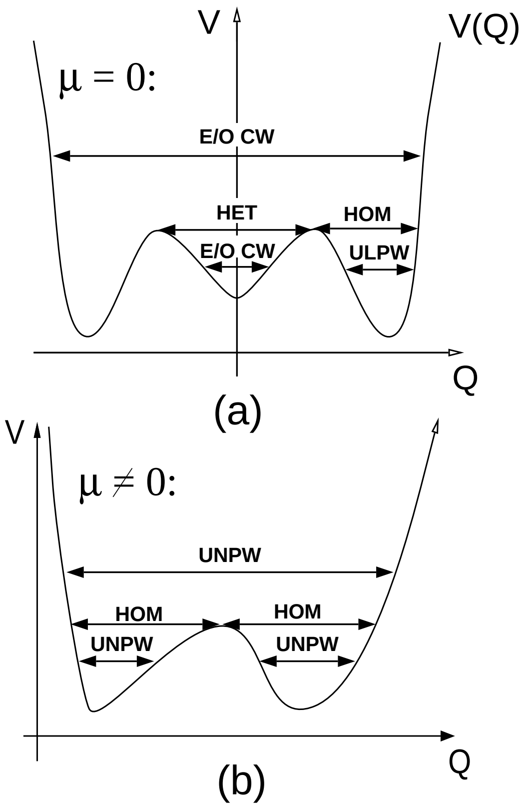

Fig. 1. (a) Possible solutions for $\mu = 0$: E/O CW-even/odd centered waves, ULPW-uncentered linear phase waves. (b) Possible solutions for $\mu \neq 0$: UNPW-uncentered nonlinear phase waves. Also shown are various homoclinic (HOM) and heteroclinic orbits (HET).

phase waves (labeled UNPW in Figure 1b). For all uncentered waves, the asymmetry of the local potential well in which an uncentered orbit lies implies that the functions Q and e^{iS} must both be periodic with the same period in order that A is periodic. Thus, for fixed values of γ and μ, we only consider periodic orbits in Q having energies β such that Q makes an integer m number of closed orbits over the interval T in z. We will call m the 'modulation number'. Given such a Q, the normalized phase profile over the real line, from (2.7), is given by

$$S(z) = \int_0^z \left(\frac{c}{2} + \frac{\mu}{Q^2(z')} \right) dz' \,. \tag{2.16}$$

Periodicity of e^{iS} requires that $S(T) = 2\pi n$, where the n is an integer, and in this case is the phase winding number. Thus we have

$$c = \frac{4\pi n}{T} - \frac{2\mu}{T} \int_0^T \frac{dz}{Q^2(z)} \,, \tag{2.17}$$

wherein the right hand side reduces to $4\pi n/T$ in the case of uncentered linear phase waves ($\mu = 0$). Note that changing the sign of μ leaves the potential (2.13) invariant, and therefore leaves Q invariant, but does change S. Hence, for nonlinear phase waves, for each periodic profile Q and choice of n there are always two traveling waves with distinct speeds.

If the function Q is symmetric about the origin, we call the corresponding wave a 'centered linear phase' wave (recall that $\mu = 0$ in this case). Due to the symmetry, we can have periodic centered waves for both cases in which Q executes an even or an odd number of half orbits across the potential well. To characterize both even and odd types of solutions for fixed γ at once, we need to consider only those symmetric orbits in Q with energies $\beta = \beta(\gamma, \mu, m, T)$ that make an integer m number of *half* orbits over the interval T in z (we will still call m the 'modulation number', although the number of full periods of Q is really $m/2$). If m is odd, then periodicity of A requires that $S(T) = n\pi$, where n is an odd integer. Likewise if m is even, then n must be even. For both cases, we have from equation (2.15) that

$$c = \frac{2\pi n}{T}. \tag{2.18}$$

Because there are no centered nonlinear phase waves, we just say that there are even and odd centered waves (labeled E/O CW in Figure 2.13). Note that in this case the term 'phase winding number' does not make sense, because Q passes through the origin. Hence we will call the integer n the 'phase number' for all the waves described above, where it is understood that this is the winding number only for the case of uncentered waves.

Note that for many possible choices of the function $h'(\cdot)$ in the potential $V(Q)$ there will be values of γ, μ and β which yield homoclinic or heteroclinic solutions (labeled HOM and HET, respectively, in Figure 1). These profiles, which are associated with solitary wave solutions, naturally partition the potential into different wells and the periodic waves into different groups associated with each well. Aside from their role in partitioning the periodic waves into distinct groups, we do not consider these waves further in this paper. We have the following definition.

Definition 2.1 *For the purposes of this paper, we define a* **family** *of periodic traveling waves to be traveling wave solutions with a particular period T and profiles Q associated with a specific well of the potential, where the wells are partitioned in the sense discussed above by heteroclinic or homoclinic profiles. Furthermore, a particular family will mean a particular choice of even or odd type (for centered waves), and with a particular modulation number m and phase number n as defined above. Finally, a family will have a specific sign of μ: strictly positive, zero, or strictly negative.*

At the beginning of this section, we argued that traveling waves should in general be characterized by three continuous parameters (one of which was the period) and two discrete parameters. In the present context, this can also be seen from the fact that for fixed T, n and m, (which fixes one of the continuous parameters and both the discrete parameters) then if any two of the remaining set of parameters $\{\mu, \gamma, \beta\}$ are chosen, the periodicity condition constrains the values of the third parameter and also the speed c. Therefore, the values of $\{\mu, \gamma, \beta\}$ for a given family (in the sense defined above) should lie on two-dimensional surfaces in the space (μ, γ, β), so that each family should be internally parameterizable by two parameters, say called μ_1 and μ_2. We will give a particularly convenient choice for μ_1 and μ_2 below.

For linear phase waves we have that $\mu = 0$, so that these families are typically parameterized by one parameter. say μ_2 (the choice of 2 in the subscript reflects the fact that during some part of our analysis later on we will choose to identify the parameter μ_1 with the integration constant μ). In this case the families lie on one-dimensional curves in the space (γ, β). For uncentered waves, each of these curves forms the the mutual boundary of two two-dimensional surfaces associated with two families of uncentered nonlinear phase waves which differ only in the sign of μ and phase profile.

We now introduce a convenient choice for μ_1 and μ_2. This choice will be especially convenient when we later change to an action-angle representation of the wave amplitude profiles.

Proposition 2.1 *A nonlinear phase wave is uniquely specified by its modulation number m, its phase winding number n, its minimum and maximum amplitudes Q_1 and Q_2, and its period T. That is, specifying these parameters uniquely determines the energy η, the momentum μ, and the parameter γ. The same characterization holds for linear phase waves, except that in this case only Q_1 or Q_2 is specified.*

Note that this does not mean that a nonlinear phase wave exists for arbitrary choices of these parameters. On the other hand, and more importantly for our purposes, the points corresponding to nonlinear phase waves always form open sets in the space (Q_1, Q_2, T), as the following proposition demonstrates. This means that the parameters Q_1, Q_2, and T may be considered to be independent variables *locally*. This also means that the parmaters μ_1 and μ_2 *always* exist locally.

Proposition 2.2 *Fix n and m. For each finite period nonlinear phase wave with associated parameters (Q_1^*, Q_2^*, T^*) there is an open neighborhood N in the space (Q_1, Q_2, T) containing the point (Q_1^*, Q_2^*, T^*), in which each point corresponds to a nonlinear phase wave with modulation number m, winding number n, and period T. Moreover, as real analytic functions, the amplitude* *and phase profiles of these waves depend smoothly on* (Q_1, Q_2, T).

Proof of Propositions 2.1 and 2.2: We prove these lemmas by first showing that the traveling waves are uniquely specified by the parameters Q_1, Q_2, and γ (with n and m fixed), secondly that traveling waves exist and depend continuously (Q_1, Q_2, γ) in some neighborhood of the point (Q_1^*, Q_2^*, γ^*), and then finally showing that the transformation from the variables (Q_1, Q_2, γ) to (Q_1, Q_2, T) is nonsingular everwhere in some neighborhood.

We begin by defining

$$W \equiv Q^2,$$

so that $W_1 = (Q_1)^2$ and $W_2 = (Q_2)^2$, and also

$$f(W; \eta, \mu^2) \equiv -\mu^2 + \eta W - \gamma W^2 + h(W)W. \tag{2.19}$$

The wave period is then given by

$$T = m \int_{W_1}^{W_2} \frac{dW}{\sqrt{f(W; \eta, \mu^2)}}. \tag{2.20}$$

Suppose that a nonlinear phase wave exists with the parameters $W_1 = W_1^*$, $W_2 = W_2^*$, $T = T^*$, $\gamma = \gamma^*$, $\eta = \eta^*$, and $\mu = \mu^*$. This assumption implies that $f(W; \eta^*, (\mu^*)^2)$ has two adjacent, simple, roots at $W_1 = W_1^*$ and $W_2 = W_2^*$ and is positive on the interval (W_1^*, W_2^*).

Due to the linear dependence of f on η, μ and γ, it can be shown that specifying W_1, W_2 and γ uniquely determines η and μ (and hence T when m is specified). Indeed, that fact that f has two adjacent simple roots at W_1 and W_2 leads to

$$-\mu^2 = -\gamma W_1 W_2 + W_1 W_2 \frac{h(W_2) - h(W_1)}{W_2 - W_1} \tag{2.21}$$

and

$$\eta = \gamma(W_1 + W_2) - \frac{h(W_2)W_2 - h(W_1)W_1}{W_2 - W_1}, \tag{2.22}$$

whereby $f(W; \eta, \gamma)$ can be written as

$$f(W; \eta, \mu^2) = \tag{2.23}$$

$$\gamma(W_2 - W)(W - W_1) + h(W)W - \frac{W_2 - W}{W_2 - W_1}h(W_1)W_1 - \frac{W - W_1}{W_2 - W_1}h(W_2)W_2.$$

From the analyticity of equations (2.21) and (2.22) for $W_1 \neq W_2$, and the fact that $\mu^2 > 0$ for all nonlinear phase waves, it follows that $f(W; \eta, \mu^2)$ is a bounded, analytic, function for each point in some open neighborhood N containing (W_1^*, W_2^*, γ^*) in the space (W_1, W_2, γ). The simple roots and boundedness of f implies that a nonlinear phase wave with finite period also corresponds to each point in N. Because f depends analytically on (W_1, W_2, γ), then so do the wave profiles.

Finally, the fact that the factor $(W_2 - W)(W - W_1)$ in equation (2.24) is positive on (W_1, W_2) implies that

$$\frac{\partial T}{\partial \gamma} < 0 \tag{2.24}$$

at each point in N, where the derivative is taken with respect to holding W_1 and W_2 constant. Thus by the implicit function theorem the transformation from (W_1, W_2, γ) to (W_1, W_2, T) is nonsingular everywhere in N.

Propositions 2.1 and 2.2 also immediately imply the following proposition:

Proposition 2.3 *Each family of nonlinear phase waves is parameterized smoothly by a single coordinate chart mapping an open set in the variables Q_1 and Q_2 onto the family of traveling waves. Likewise, each family of linear phase waves is parameterized smoothly by a single coordinate chart in the variable Q_1 (or Q_2).*

This concludes the discussion of GNLS wave classification. We now discuss the relationship of the general traveling waves to the so-called rotating wave solutions. Rotating waves are the simplest type of traveling waves, having the form

$$A^{(n)}(x,t) = a\exp(i(2\pi nx - \omega_n t + \phi))\,, \tag{2.25}$$

where a is a real positive constant, $\omega_n = 4\pi^2 n^2 + h'(|a|^2)$, and ϕ is a trivial (real) phase parameter. The Q profiles of these solutions lie exactly at the local extrema of the potential (2.13), and therefore are not parameterized by a modulation number m (because Q is constant). In fact, the explicit solution (2.25) shows that the rotating waves are (up to the trivial phase) only parameterized by one discrete parameter n and one continuous parameter a.

If we allow m and n to varying continuously, which is equivalent to relaxing the periodic boundary conditions, then all rotating wave solutions located at the minima of the potential $V(Q)$ (excluding the trivial solution) may be viewed as continuous limits of families of uncentered traveling waves. This limit is degenerate in the sense that a given rotating wave with phase number n will be a limiting case for all families of uncentered traveling waves possessing the same phase number n but different (all) values of the modulation number m (where $m > 0$).

When periodic boundary conditions are imposed, the number of uncentered traveling wave solutions is greatly reduced through the quantization expressed by the integers m and n. The families of rotating waves are less reduced, in a sense, because they are not parameterized by m to begin with. The result is that not all rotating waves with a given period are continuous limits of traveling wave families of the same period. Intuitively one expects that if a rotating wave with period T and phase number n is neutrally stable with respect to sideband perturbations (defined below) of wavenumber $k_m = 2\pi m/T$, then, to leading order, perturbations of the rotating wave in the k_m direction will generate uncentered traveling waves with period T and with parameters m and n, i.e., the rotating wave is the limit of such a family of traveling waves. This situation is analogous to the foliation of periodic orbits surrounding elliptic fixed points in two-dimensional area-preserving flows. Conversely, one expects that this would not be possible if the rotating wave was modulationally unstable in the k_m direction.

To show that this is precisely the case, we first calculate the linearized stability of the rotating waves in the unperturbed GNLS equation (we specialize to $T = 1$ here without loss of generality). Following standard linearized stability theory (see [8] for example), a sideband perturbation with wavenumber $k_m = 2\pi m$ of the rotating wave solution $A^{(n)}(x,t)$ (where $A^{(n)}(x,t)$ has the form (2.25)) is defined to be

$$A(x,t) = A^{(n)}(x,t)(1 + p^+(t)e^{ik_m x} + p^-(t)e^{-ik_m x}), \tag{2.26}$$

where $p^+(t)$ and $p^-(t)$ are the (small) complex perturbation amplitudes. Substituting (2.26) into equation (1.1) and keeping only the linear terms, the following system for the evolution of $p^+(t)$ and $p^-(t)$ is derived,

$$\partial_t \begin{pmatrix} p^+ \\ p^{-*} \end{pmatrix} = \begin{pmatrix} -C^+ & -ih''(a^2)a^2 \\ ih''(a^2)a^2 & -C^{-*} \end{pmatrix} \begin{pmatrix} p^+ \\ p^{-*} \end{pmatrix}, \tag{2.27}$$

where

$$C^\pm = i(\pm 2k_n k_m + k_m^2) + ih''(a^2)a^2\,,$$

and where $k_n = 2\pi n$. The eigenvalues of the matrix in system (2.27) are given by

$$\lambda^{\pm} = ik_m \left(2k_n \pm \sqrt{k_m^2 + 2h''(a^2)a^2}\right) . \tag{2.28}$$

It follows that for any function $h(\xi)$ convex for $\xi \geq 0$, all the rotating waves are linearly neutrally stable. On the other hand, if $h''(a^2) < 0$ then the number of unstable modes possessed by the rotating wave of amplitude a is exactly

$$2\left[\frac{a}{\pi}\sqrt{\frac{-h''(a^2)}{2}}\right] , \tag{2.29}$$

where the square brackets give the greatest integer less than the number between them.

To prove that any neutrally stable rotating wave $A^{(n)}(x,t)$ is the limit of families of all uncentered traveling waves with indices m and n, we may take advantage of the eigenvalue expression (2.28) above, which shows that the n^{th} rotating wave is neutrally stable to the m^{th} sideband if and only if the inequality

$$k_m^2 + 2h''(a^2)a^2 > 0 \tag{2.30}$$

is satisfied. The necessary and sufficient condition for a family of traveling waves with parameters m and n to reduce smoothly to a rotating wave with parameters n and a is that the frequency of oscillations of Q in the potential $V(Q)$ (equation (2.13)) linearized around the minima of the potential approaches k_m as $Q \to a$, or

$$\lim_{Q\to a}\sqrt{\frac{V''(Q)}{2}} = \sqrt{\frac{V''(a)}{2}} = k_m . \tag{2.31}$$

From (2.13), we have

$$V''(a) = -8\gamma - 4h''(a^2)a^2 - 8h'(a^2) , \tag{2.32}$$

where we have used the extrema condition $V'(a) = 0$ to eliminate μ. Writing a rotating wave in the form

$$A^{(n)}(x,t) = ae^{-i\alpha t + ik_n z} , \tag{2.33}$$

and substituting this into the profile equation (2.2), we also have that for rotating waves,

$$\gamma = \frac{c^2}{4} + k_n^2 - ck_n + h'(a^2)\,.$$

Upon substituting this into (2.32), and the resulting expression into condition (2.31), condition (2.31) becomes

$$(c - 2k_n)^2 = k_m^2 + 2h''(a^2)a^2\,.$$

Because the left side is always nonnegative, it can now be seen that this condition cannot be satisfied unless the neutral stability criteria (2.30) is also met. If the stability criteria is not met, the rotating wave possesses local hyperbolic structure in the k_m direction (stable and unstable manifolds), which may be associated with types of solutions different from traveling waves, such as homoclinic orbits.

3 Criteria for Persistence of Uncentered Traveling Wave Solutions

In the preceding section we classified the periodic traveling wave solutions of the GNLS equation, which by definition in this paper are the form (1.3), where S and Q are real-valued functions. These solutions were classified into even/odd centered waves and uncentered linear/nonlinear phase waves. In this section we describe necessary criteria for the persistence of these traveling waves under the GCGL perturbation. We also give sufficient criteria for the persistence of uncentered waves. Sufficient criteria for centered waves are given in the next section.

Recall that in the preceding section we found that each family of uncentered periodic traveling wave solutions of the GNLS equation of period T can be (generically) parameterized by two real continuous parameters μ_1 and μ_2, and two integers $m > 0$ and n, where the 'modulation number' m is the number of oscillations of Q per period T and 'phase number' n is the phase winding number. For each choice of m and n, the Melnikov criteria (1.4) and (1.5) will yield two equations in the parameters μ_1 and μ_2, and will therefore generically hold only for a discrete subset of traveling waves of a given period. It will be verified in the case study in the third paper of this series [2] that this is the case, which will also relate the discrete subset of persisting waves to bifurcations involving rotating waves.

On the other hand, families of linear phase waves are parameterized by a modulation number m and phase number n and only one continuous parameter, say μ_2, so that we would not expect the Melnikov criteria to be satisfied by any μ_2 in general. However, it will turn out that for $n = 0$ a certain linear combination of the Melnikov conditions is satisfied identically for all μ_2, leading again to the persistence of a discrete subset of traveling waves of a given period. Proofs of the existence of these discretely selected waves and their connections to bifurcations involving rotating waves are also given in [2].

We now inquire after *sufficient* criteria for traveling waves to persist. Before stating some propositions, we will first rewrite the equations in another form (to simplify the definition of Poincaré sections later) and define some assumptions and special coordinates which will be used in the statement of the propositions.

For traveling waves of the form (2.1) of the full GCGL equation, B is a complex-valued function of $z = x - ct$ that satisfies the profile equation

$$(i + \epsilon)\partial_{zz}B + i\alpha B + c\partial_z B - ih'(|B|^2)B - \epsilon g'(|B|^2)B = 0\,. \tag{3.1}$$

The question of the existence of periodic traveling waves for the GCGL equation reduces to the question of the existence of periodic solutions of this equation.

Letting $B = Qe^{iS}$ and moving all the terms involving ϵ to the right hand side, equation (3.1) becomes

$$\begin{aligned}&\partial_{zz}Q - Q(\partial_z S)^2 + 2iQ_z\partial_z S + iQ\partial_{zz}S - ic\partial_z Q + cQ\partial_z S + \alpha Q - h'(Q^2)Q \\ &= \epsilon\frac{\epsilon - i}{1+\epsilon^2}\left(-ic\partial_z Q + cQ\partial_z S + \alpha Q - h'(Q^2)Q + g'(Q^2)Q\right)\,.\end{aligned} \tag{3.2}$$

Introducing the new variable

$$P \equiv Q^2(\partial_z S - c/2)\,, \tag{3.3}$$

we change from a polar form representation to an amplitude and momentum representation. Note that P is constant when $\epsilon = 0$ and reduces to the constant μ by equation (2.6).

Separating real and imaginary parts, equation (3.2) becomes the system

$$\begin{aligned}&\partial_{zz}Q - \frac{P^2}{Q^3} + \left(\alpha + \frac{c^2}{4}\right)Q - h'(Q^2)Q \qquad (3.4)\\ &= -\frac{\epsilon}{1+\epsilon^2}cQ_z + \frac{\epsilon^2}{1+\epsilon^2}\left(c\frac{P}{Q} + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2)\right)Q\right)\,,\end{aligned}$$

$$\partial_z P = -\frac{\epsilon}{1+\epsilon^2}\left(cP + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2)\right)Q^2\right) - \frac{\epsilon^2}{1+\epsilon^2} cQ\partial_z Q \,. \tag{3.5}$$

Notice that when $\epsilon = 0$ this system reduces to the GNLS traveling wave profile system (2.8).

We now write these equations as a perturbation of the Hamiltonian system (2.10–2.13) for the GNLS traveling waves by introducing the variable

$$R \equiv \partial_z Q \,.$$

The profile system becomes

$$\partial_z P = -\frac{\epsilon}{1+\epsilon^2}\left(cP + \left(\frac{c^2}{4} + \gamma - h'(Q^2) + g'(Q^2)\right)Q^2\right) - \frac{\epsilon^2}{1+\epsilon^2} cQR \,, \tag{3.6}$$

$$\partial_z Q = \frac{\partial H}{\partial R} \,, \tag{3.7}$$

$$\partial_z R = -\frac{\partial H}{\partial Q} - \frac{\epsilon}{1+\epsilon^2} cR + \frac{\epsilon^2}{1+\epsilon^2}\left(c\frac{P}{Q} + \left(\frac{c^2}{4} + \gamma - h'(Q^2) + g'(Q^2)\right)Q\right) \,, \tag{3.8}$$

where H and the potential V are given by

$$H \equiv \frac{1}{2}\left(R^2 + V(Q, P; \gamma)\right) \,, \quad V(Q, P; \gamma) \equiv \frac{P^2}{Q^2} + \gamma Q^2 - h(Q^2) \,, \tag{3.9}$$

respectively, and where

$$\gamma \equiv \frac{c^2}{4} + \alpha \,. \tag{3.10}$$

Notice that the perturbation adds an extra dimension to the Hamiltonian system of the traveling wave for the GNLS equation.

Finally, for each family of traveling waves of the unperturbed system in the sense defined by definition (2.1), we define a final change of coordinates from (P, Q, R) to the action-angle variables (P, I, Θ) of the unperturbed system, where I and Θ are defined by

$$P = P \tag{3.11}$$

$$I \equiv \frac{1}{\pi} \int_{Q^{(1)}}^{Q^{(2)}} \sqrt{2H - V(Q', P; \gamma)}\, dQ' , \tag{3.12}$$

$$\Theta \equiv \frac{2\pi}{F(I, P, \gamma)} \int_{Q^{(1)}}^{Q} \frac{dQ'}{\sqrt{2H - V(Q', P; \gamma)}} , \tag{3.13}$$

where H is the energy of the orbit determined by P, Q and R and γ, and where $Q^{(1)}(P, Q, R, \gamma)$ and $Q^{(2)}(P, Q, R, \gamma)$ are any two consecutive zeros of $2H - V(Q, P; \gamma)$ such that $2H - V(Q, P; \gamma)$ is positive for $Q^{(1)} < Q < Q^{(2)}$. The variable $F(I, P, \gamma)$ is the fundamental period of the profile Q (the symbol T being reserved for the full traveling wave period, which may be a multiple of F) given by

$$F(I, P, \gamma) \equiv 2 \int_{Q^{(1)}}^{Q^{(2)}} \frac{dQ'}{\sqrt{2H(I) - V(Q', P; \gamma)}} . \tag{3.14}$$

Inverting these equations, we can in principle obtain the inverse transformation

$$\begin{aligned} Q &= Q(P, I, \Theta, \gamma) , \\ R &= R(P, I, \Theta, \gamma) , \end{aligned} \tag{3.15}$$

which the reader should note depends only on the structure of the unperturbed system.

With the above definitions, the perturbed profile system becomes

$$\partial_z P = \epsilon\, f_1(P, I, \Theta, \gamma, c, \epsilon) , \tag{3.16}$$
$$\partial_z I = \epsilon\, f_2(P, I, \Theta, \gamma, c, \epsilon) , \tag{3.17}$$
$$\partial_z \Theta = \Omega(P, I, \gamma) + \epsilon\, f_3(P, I, \Theta, \gamma, c, \epsilon) , \tag{3.18}$$

where $\Omega \equiv 1/F$ and

$$f_1 = -\frac{1}{1+\epsilon^2} \left(cP + \left(\frac{c^2}{4} + \gamma - h'(Q^2) + g'(Q^2) \right) Q^2 \right) - \frac{\epsilon}{1+\epsilon^2} cQR , \tag{3.19}$$

$$\begin{aligned} f_2 = &-\frac{1}{1+\epsilon^2} \left(\frac{\partial I}{\partial P} \left(cP + \left(\frac{c^2}{4} + \gamma - h'(Q^2) + g'(Q^2) \right) Q^2 \right) + \frac{\partial I}{\partial R} cR \right) \\ &+ \frac{\epsilon}{1+\epsilon^2} \Big(- \frac{\partial I}{\partial P} cQR \end{aligned}$$

$$+\frac{\partial I}{\partial R}\left(c\frac{P}{Q}+\left(\frac{c^2}{4}+\gamma-h'(Q^2)+g'(Q^2)\right)Q\right)\Bigg), \quad (3.20)$$

$$f_3=-\frac{1}{1+\epsilon^2}\left(\frac{\partial\Theta}{\partial P}\left(cP+\left(\frac{c^2}{4}+\gamma-h'(Q^2)+g'(Q^2)\right)Q^2\right)+\frac{\partial\Theta}{\partial R}cR\right)$$
$$+\frac{\epsilon}{1+\epsilon^2}\Bigg(-\frac{\partial\Theta}{\partial P}cQR$$
$$+\frac{\partial\Theta}{\partial R}\left(c\frac{P}{Q}+\left(\frac{c^2}{4}+\gamma-h'(Q^2)+g'(Q^2)\right)Q\right)\Bigg). \quad (3.21)$$

In Proposition 2.3 we established that each family of GNLS nonlinear phase waves is smoothly parameterized by two parameters μ_1 and μ_2 (which may be taken to be the minimum and maximum amplitudes Q_1 and Q_2). In the unperturbed action-angle system, this is manifested by the fact, which we shall prove shortly, that families of unperturbed nonlinear phase waves correspond to smooth two-dimensional surfaces in the space of variables consisting of the initial conditions P_0 and I_0 (Θ_0 being arbitrary) and the parameter γ. Because a family has a fixed period T and modulation number m (by definition) these surfaces are determined by the equation $\Omega(I_0, P_0, \gamma) = m/T$. We will denote such a surface by the symbol Σ_T.

To consider the peristence of orbits under perturbation, we will consider surfaces of section for the system (3.16)-(3.18). It might seem natural to use the variables P_0 and I_0 on these surfaces of section. We would also like to work with families of traveling waves, i.e. waves all of the same period. However, the surfaces Σ_T (on which the families lie) may not neccessarily be embedded in the space (P_0, I_0, γ) in such a way that they can be covered with a single coordinate chart in any two of the variables P_0, I_0 and γ, i.e. , a coordinatization of Σ_T by any two of these variables may contain singularities. Therefore, we will set up the analysis in the general coordinates μ_1 and μ_2 which will parameterize any given surface Σ_T. We now establish the existence of such coordinates (by way of example).

Recall Proposition 2.2, which stated that for fixed n and m, for each nonlinear phase wave with associated parameters (Q_1^*, Q_2^*, T^*) there is an open neighborhood N in the space (Q_1, Q_2, T) containing the point (Q_1^*, Q_2^*, T^*), in which each point corresponds to a nonlinear phase wave with modulation number m, winding number n, and period T. Moveover, these waves depended analytically on (Q_1, Q_2, T). We now prove the following proposition:

Proposition 3.1 *For fixed n and m, for each nonlinear phase wave with associated parameters (Q_1^*, Q_2^*, T^*) there is an open neighborhood M in the space (Q_1, Q_2, T) containing the point (Q_1^*, Q_2^*, T^*), in which the action-angle transformation from (Q_1, Q_2, T) to (P, γ, I) generated by the wave profiles associated with each point in M, is nonsingular.*

In other words, the action-angle transformation does not introduce any new singularities.

Proof of the proposition: Let $W_1 = Q_1^2$, $W_2 = Q_2^2$. From the fact the transformation from (Q_1, Q_2, T) to (Q_1, Q_2, γ) is nonsingular (from the proof of Propositions 2.1 and 2.2) it follows that the transformation is nonsingular if and only if

$$P_{W_1} I_{W_2} - P_{W_2} I_{W_1} \neq 0, \tag{3.22}$$

where the subscripts denote partial derivatives with respect to the variables W_1, W_2, and γ.

Recall the definition

$$f(W; \eta, \mu^2) \equiv -\mu^2 + \eta W - \gamma W^2 + h(W)W, \tag{3.23}$$

and also that the condition for a nonlinear phase wave to exist in the first place at is that $f(W; \eta, \mu^2)$ has two adjacent, simple, adjacent, roots at W_1 and W_2 and is positive on the interval (W_1, W_2).

To determine the signs of the partial derivatives in equation (3.22), we now consider variations in F with respect to μ^2 and η^2. We define

$$\Delta f(W; \Delta\eta, \Delta\mu^2) \equiv -\Delta\mu^2 + \Delta\eta W,$$

so that

$$f(W; \eta + \Delta\eta, \mu^2 + \Delta\mu^2) = f(W; \eta, \mu^2) + \Delta f(W; \Delta\eta, \Delta\mu^2).$$

Now consider variations $\Delta\mu^2$ and $\Delta\eta$ that leave the root W_2 invariant but produce a varition ΔW_1 in the root W_1. To keep the root W_2 invariant, we require

$$f(W_2; \eta + \Delta\eta, \mu^2 + \Delta\mu^2) = 0 = f(W_2, \eta, \mu^2) + \Delta f(W_2; \Delta\mu^2, \Delta\eta) = 0 - \Delta\mu^2 + \Delta\eta W_2,$$

or

$$\Delta\eta = \frac{\Delta\mu^2}{W_2}. \tag{3.24}$$

That f possesses a root at $W_1 + \Delta W_1$ implies that

$$f(W_1 + \Delta W_1; \eta, \mu^2) + \Delta f(W_1 + \Delta W_1; \Delta\eta, \Delta\mu^2) = 0,$$

or

$$f(W_1+\Delta W_1;\eta,\mu^2)+-\Delta\mu^2+\Delta\eta W_1+\Delta\eta\Delta W_1=0.$$

Using (3.24), this becomes

$$\Delta\mu^2\left(\frac{W_1}{W_2}-1+\frac{\Delta W_1}{W_2}\right)=f(W_1+\Delta W_1;\eta,\mu^2).$$

Dividing this by ΔW_1 and taking the limit as $\Delta W_1\to 0$ we obtain

$$\frac{\partial\mu^2}{\partial W_1}=\frac{\frac{\partial f}{\partial W}(W_1)}{1-\frac{W_1}{W_2}}, \tag{3.25}$$

where the partial derivative signifies differentiation with respect to W_1 keeping W_2 and γ constant. Because $\partial f/\partial W(W_1)$ is strictly postive, and because P is a monotonically increasing function of μ^2 ($P=\mu$) and $(1-\frac{W_1}{W_2})>0$, we immediately have that $P_{W_1}>0$.

A similar argument establishes that $P_{W_2}>0$. In terms of W_1 and W_2, the action I becomes

$$I=\frac{1}{2\pi}m\int_{W_1}^{W_2}\frac{1}{W}\sqrt{f(W;\eta,\mu^2)}. \tag{3.26}$$

For variations $\Delta\mu^2$ and $\Delta\eta$ that leave the root W_2 invariant but produce a varition ΔW_1 in the root W_1, equation (3.24) implies that

$$\Delta f(W;\Delta\eta,\Delta\mu^2)=-\Delta\mu^2+\Delta\eta W=\Delta\mu^2(-1+\frac{W}{W_2})<0,$$

for all W in (W_1,W_2). This, along with the fact that $f(W;\eta,\mu^2)$ vanishes at W_1 and W_2, implies that $I_{W_1}<0$. A similar argument establishes that $I_{W_2}>0$.

From the inequalities derived above, we conclude that the left hand side of (3.22) is strictly positive, and hence that the transformation from (W_1,W_2,γ) to (P,γ,I) is nonsingular at each nonlinear phase wave with finite period. This concludes the proof.

We can now summarize all that we need to know about the surfaces Σ_T:

Proposition 3.2 *For each nonlinear phase wave, corresponding to a point* (P^*,I^*,γ^*)*, there is an open neighborhood* N *containing this point in the space*

(P, I, γ) in which each point corresponds to a nonlinear phase wave with the same modulation number m and winding number n. Moreover, as real analytic functions, the amplitude and phase profiles and the periods of these waves depend smoothly on (P, I, γ). Finally, this neighborhood may also be smoothly and nonsingularly parameterized by three variables μ_1, μ_2 and T, where μ_1 and μ_2 may be taken to be the minimum and maximum wave amplitudes Q_1 and Q_2, respectively, and T is the wave period. In other words, N is foliated by two-dimensional surfaces Σ_T of constant period.

Proof: The proof is a direct consequence of Propositions 3.1, 2.1, and 2.2).

We will from now on assume that m and n are fixed and that we are working within a specific neighborhood N of a traveling wave in which the transformation from the variables (P_0, γ, I_0) to (μ_1, μ_2, T) is fixed and nonsingular, which the propositions above gaurantee to be always possible. To alleviate any doubts about which variables are independent and which are not, we now explicitly choose the four variables μ_1, μ_2, T, and the wave speed c to be independent. These choices will hold for the perturbed system as well, in which case T will still have the meaning of the period of the *unperturbed* orbit associated with the initial data (μ_1, μ_2, T). By choosing μ_1, μ_2, T, and the wave speed c to be independent, the parameter γ is necessarily constrained, i.e. given by the some function $\gamma(\mu_1, \mu_2, T, m)$ obtained by inverting the transformation from (P_0, γ, I_0) to (μ_1, μ_2, T), and the frequency α is constrained to be

$$\alpha = \alpha(\mu_1, \mu_2, T, c, m) = \gamma(\mu_1, \mu_2, T, m) - c^2/4 \,. \tag{3.27}$$

We henceforth suppress the dependence on m in these and similar expressions.

For arbitrary choices of initial data (μ_1, μ_2, T, c) (note that c must be included in the initial data if $\epsilon > 0$) the corresponding unperturbed traveling wave will generally *not* be periodic because the phase S will not generally have integer winding number. But from the fact that the potential $V(Q, P; \gamma)$ depends only on the parameters γ and P (P being a constant of integration in the unperturbed dynamics), it follows that any given choice of μ_1, μ_2, T, and m completely determines the wave amplitude profile $Q = Q(\mu_1, \mu_2, T; z)$ and the momentum variable $P = P(\mu_1, \mu_2, T)$. Therefore, from (2.16), for an arbitrary value of c the unperturbed phase profile S is given by

$$S(\mu_1, \mu_2, T, c; z) = \int_0^z \left(\frac{c}{2} + \frac{P(\mu_1, \mu_2, T; z')}{Q(\mu_1, \mu_2, T; z')^2} \right) dz' \,. \tag{3.28}$$

It follows from this that there exists a unique function $c = c(\mu_1, \mu_2, T)$ for which the traveling wave is periodic with winding number n, i.e. $c(\mu_1, \mu_2, T)$ solves $S(\mu_1, \mu_2, T, c(\mu_1, \mu_2, T); T) = 2\pi n$. Thus the initial data (μ_1, μ_2, T, c) for unperturbed waves is fully specified by just the triplet (μ_1, μ_2, T).

We can now pose precisely the question of persistence: Consider a particular periodic uncentered traveling wave A_* of the GNLS equation with modulation/phase numbers m and n, period T_*, and corresponding initial data $(\mu_{1*}, \mu_{2*}, T_*, c(\mu_{1*}, \mu_{2*}, T_*))$. Does A_* persist as a period T_* traveling wave of the GNLS equation with parameters m and n? More precisely, does there exist a unique periodic uncentered traveling wave A_ϵ of the GCGL equation with parameters m and n and period T_* for all ϵ less than some $\epsilon_0 > 0$ and corresponding initial data $(\mu_{1*}(\epsilon), \mu_{2*}(\epsilon), T(\epsilon), c(\epsilon))$ such that $\lim_{\epsilon \to 0}(\mu_{1*}(\epsilon), \mu_{2*}(\epsilon), T(\epsilon), c(\epsilon)) = (\mu_{1*}, \mu_{2*}, T_*, c_*)$?

To phrase an answer to this question, we present the Melnikov functions (derived later)

$$M^1(\mu_1, \mu_2, T, c) \equiv \int_0^T \left(cP + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2) \right) Q^2 \right) dz\,, \qquad (3.29)$$

$$M^2(\mu_1, \mu_2, T, c) \equiv \int_0^T \left(c\frac{P^2}{Q^2} + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2) \right) P + cR^2 \right) dz\,, \qquad (3.30)$$

where Q, P are the solutions $Q(\mu_1, \mu_2, T; z)$ and $P(\mu_1, \mu_2, T)$ of the unperturbed system and with $\alpha = \alpha(\mu_1, \mu_2, T, c)$ as defined by (3.27). These functions are derived in appendix A directly from the system (3.16–3.18). Using the relationships between P, Q and S in the unperturbed system obtained from (3.3) and by setting $\epsilon = 0$ in (3.4), and integrating by parts, it is easy to show that these Melnikov functions are linear combinations of, and are therefore equivalent to, the Melnikov functions (1.4) and (1.5), derived by averaging.

We have the following proposition:

Proposition 3.3 *(i) Given a candidate periodic traveling wave A_* of either type (centered or uncentered) with initial data $(\mu_{1*}, \mu_{2*}, T_*, c_*)$ and modulation/phase numbers m and n, a necessary condition for A_* to persist as a periodic wave with the same period T_* and modulation/phase numbers m and n is*

$$M^1(\mu_{1*}, \mu_{2*}, T_*, c_*) = M^2(\mu_{1*}, \mu_{2*}, T_*, c_*) = 0\,. \qquad (3.31)$$

(ii) Supposing that (i) is satisfied, and additionally that A_* is an uncentered wave such that assumptions H_1 and H_2 hold, a sufficient condition for A_* to persist as an uncentered periodic wave with period T_* and parameters

m and n is

$$\det\begin{pmatrix} M^1_{\mu_1} & M^1_{\mu_2} & M^1_c \\ M^2_{\mu_1} & M^2_{\mu_2} & M^2_c \\ S_{\mu_1} & S_{\mu_2} & \frac{1}{2} \end{pmatrix}\Bigg|_{(\mu_{1*},\mu_{2*},T_*,c_*)} \neq 0\,. \tag{3.32}$$

The proof of this proposition is given in appendix A.

In practice, we are mostly concerned with finding waves for which the necessary conditions are met, because this turns out to be the main obstruction to persistence. The sufficient condition for uncentered waves is met generically, and can be checked as an afterthought.

As a practical matter, to reduce the problem to a manageable size, we will also find it useful to search for persisting waves by analyzing the properties of the Melnikov functions only on fully periodic waves. To this end we set $c = c(\mu_1, \mu_2, T)$ (the speed which yields fully periodic waves, as shown to exist through the discussion involving expression (3.28)) and define 'restricted' Melnikov functions

$$W^{(1,2)}(\mu_1, \mu_2, T) \equiv M^{(1,2)}(\mu_1, \mu_2, T, c(\mu_1, \mu_2, T))\,. \tag{3.33}$$

On periodic waves the vanishing of these functions is clearly identical to the necessary condition given above, but now we may search for persisting periodic waves by searching for points at which $W^{(1,2)}(\mu_1, \mu_2, T) = 0$ in the three-dimensional space of initial data (I_0, P_0, γ) (equivalently (μ_1, μ_2, T)), instead of the larger space (I_0, P_0, γ, c). Because there are two Melnikov conditions, the locus of points satisfying the restricted conditions will generically lie on one-dimensional curves in this space.

We may still obtain some broader information about sufficiency in the restricted space according to the following proposition:

Proposition 3.4 *For an uncentered periodic wave A_* satisfying assumptions H_1 and H_2 and the necessary condition of Proposition 3.3, a sufficient condition for A_* to persist as a nonempty set of traveling waves for which the amplitude profiles have period T_*, but with $S(T_*)$ not necessarily equal to $2\pi n$, is that*

$$\frac{\partial(W^1, W^2)}{\partial(\mu_1, \mu_2)}\bigg|_{(\mu_{1*},\mu_{2*},T_*)} \neq 0\,. \tag{3.34}$$

The proof of this proposition also appears in appendix A.

Remark 3.1 *Note that this condition is not sufficient for persistence of* fully *periodic waves. Condition (3.32) is still needed to insure that* $S(T_*) = 2\pi n$*. On the other hand, the proposition suggests a more general picture encompassing both nonperiodic and periodic waves for which* n *is a continuous parameter.*

Remark 3.2 *Geometrically, this condition means that on surfaces* Σ_T *of constant period, the restricted Melnikov functions each have roots along one-dimensional curves (which we call 'root-lines') embedded in* Σ_T *which cross transversely at the point* $(\mu_{1*}, \mu_{2*}, T_*)$*. Therefore the broader persistence problem is reduced to a two-dimensional problem of finding these crossings. Finding such transverse crossings is also the easiest way to numerically locate points at which the* necessary *conditions are satisfied.*

4 Persistence Criteria for Linear Phase Waves

For the general case of uncentered waves, it is difficult to analyze the Melnikov functions exhaustively. However, in the case of linear phase waves, simpler persistence criteria can be obtained, which we derive in this section. These criteria are analyzed thoroughly in the case study in the third paper of this series [2].

Recall that for unperturbed linear phase waves, the integration constant μ (the variable P_0 in the unperturbed dynamics) is identically zero, and the phase of the unperturbed waves is simply given by

$$S(z) = \frac{c}{2} z \, . \tag{4.1}$$

Furthermore, the speed c is just $2\pi n/T$ for centered waves and $4\pi n/T$ for uncentered linear phase waves. A special subclass of these waves are the stationary waves with $c = n = 0$, which have the simple form

$$A = Q(x) e^{-i\alpha t} \, , \tag{4.2}$$

where Q is real valued and periodic. This subclass includes both uncentered linear phase waves and even centered waves with $n = 0$ (odd centered waves always have $n \neq 0$).

We have the following proposition:

Proposition 4.1 *Only stationary* $(n = c = 0)$ *linear phase waves can per-*

sist under the GCGL perturbation (excluding rotating waves). Of these, both necessary conditions (3.31) are met if

$$M^1(A_*) \equiv \int_0^T \left(\alpha - h'(Q^2) + g'(Q^2)\right) Q^2\, dz = 0\,, \tag{4.3}$$

where $A_* = Q(x)e^{-i\alpha t}$.

Thus, among others, *all* odd centered waves fail to persist.

Proof: Setting $P = \mu = 0$ in (3.29) and (3.30) we obtain

$$M^1(A_*) = \int_0^{T_*} \left(\alpha - h'(Q^2) + g'(Q^2)\right) Q^2\, dz = 0\,, \tag{4.4}$$

$$M^2(A_*) = \int_0^{T_*} cR^2\, dz = 0\,. \tag{4.5}$$

The first condition is the same as that stated in the proposition. Since R does not vanish for all linear phase waves (except on rotating waves), then the second condition demands that $c = 0$ and hence $n = 0$. This completes the proof.

We now seek sufficient conditions for persistence. For uncentered waves, some useful information can be obtained from proposition 3.4 (Proposition 3.4 cannot be applied to centered waves, because they are only defined at $P = 0$, and hence are not embedded in a two parameter family).

To apply proposition 3.4, we assume assumption H_1 has been shown to hold for a candidate uncentered linear phase wave A_*, that is, there is a neighborhood $\mathcal{N}$ of the initial data (P_*, I_*, γ_*) in the space (P_0, I_0, γ) which can be coordinatized by local variables (μ_1, μ_2, T), where T is the period of the corresponding uncentered wave.

Now we refine the choice of μ_1 and μ_2. Note that the surfaces of constant period Σ_T in $\mathcal{N}$ are symmetric with respect to the I_0, γ plane because the integration constant μ (which equals P_0) is squared in the potential $V(Q, P; \gamma)$ (see (3.9)). This implies that we may choose local coordinates μ_1 and μ_2 such that

$$\mu_1 \equiv P_0 \tag{4.6}$$

and μ_2 is a function,

$$\mu_2 \equiv f(I_0, \gamma), \tag{4.7}$$

where f is some function independent of P_0, for a sufficiently small neighborhood $\mathcal{N}$. Recall from Remark (**??**) that each family of linear phase waves of period T is characterized by one continuous parameter. With the choice (4.6) and (4.7) of coordinates, μ_2 becomes exactly this one continuous parameter (and $\mu_1 = P_0 = 0$ for the linear phase waves). Also note that with this parameterization, the parameter γ is given in $\mathcal{N}$ by some function $\gamma(\mu_1, \mu_2, T)$ which is now an even function of μ_1.

Recall that the restricted Melnikov conditions $W^{(1,2)}(\mu_1, \mu_2, T)$ are defined by definition (3.33) by restricting the speed c in definitions (3.29) and (3.30) such that A_* is fully periodic. We have the following proposition:

Proposition 4.2 *Suppose there exists and uncentered linear phase wave A_* of the GNLS equation with period $T*$ and $n = 0$ that satisfies assumption H_1 at $(\mu_1 = 0, \mu_2 = \mu_{2*})$ where the variables (μ_1, μ_2) have been chosen according to (4.6) and (4.7). Suppose also that A_* satisfies the necessary condition for persistence of Proposition 4.1. Then a sufficient condition for A_* to persist as a traveling wave for which the amplitude profile has period T_*, but which is not necessarily fully periodic (i.e. , with $S(T_*)$ not necessarily equal to 0), is*

$$\left.\frac{\partial W^1}{\partial \mu_2}\frac{\partial W^2}{\partial \mu_1}\right|_{(0,\mu_{2*},T_*)} \neq 0\,. \tag{4.8}$$

Remark 4.1 *This proposition tells us that if the necessary conditions are satisfied, the restricted sufficiency condition (3.4) is met as long as W^1 and W^2 both vary linearly at A_* in μ_2 and μ_1, respectively. The latter condition occurs generically, so we can see that the sufficient condition is almost always met.*

Proof: The proposition follows immediately from proposition 3.4 and the fact that with the way we have defined the local variables (μ_1, μ_2) the restricted Melnikov functions W^1 and W^2 are even and odd functions of μ_1, respectively. To see this, recall that the restricted Melnikov functions (3.33) were defined by restricting the speed c to a function $c(\mu_1, \mu_2, T)$ such that the corresponding traveling wave is periodic, i.e. $S(T) = 2\pi n$. With our definition of the local variables, for an $n = 0$ linear phase uncentered wave we obtain from (2.17) that

$$c(\mu_1, \mu_2, T) = -\frac{\mu_1}{T}\int_0^T \frac{dz}{Q^2(z; \mu_1, \mu_2, T)}\,, \tag{4.9}$$

where Q is the amplitude profile associated with the initial data (μ_1, μ_2, T). Q is an even function of μ_1 because γ is. Therefore $c(\mu_1, \mu_2, T)$ is an odd function of μ_1. The even and oddness of the Melnikov functions now follows from the oddness of P_0 and c in μ_1, and the definitions (3.29) and (3.30). This completes the proof.

Proposition 4.2 does not yield information about the speed or the phase of the persisting traveling wave. Nor does it yield information about centered waves. In fact, stronger criteria can be found by taking into account certain symmetries of the governing equations at $c = 0$.

Before giving these criteria, we need to define a technical assumption analogous to assumption H_1. From the discussion of section 2 it can be seen that members of a family of centered waves of the unperturbed system have profiles Q that are completely characterized by their energy β and the parameter γ (recall that $\mu = 0$ for centered waves). Alternatively, we can specify γ and the initial condition (P_0, Q_0, R_0) for equations (3.6–3.8). For centered waves, we have $P_0 = 0$ ($\mu = 0$), and we may choose $Q_0 = 0$ (since all centered profiles pass through the R axis to avoid a singularity in (3.6–3.8)), so that the wave is specified only by γ and R_0. In a similar fashion, a linear phase uncentered wave may be completely characterized by γ and Q_0 with $P_0 = R_0 = 0$. We now define the following assumption analogous to assumption H_1:

H_3: Given a candidate uncentered (centered wave) A_*, there is open neighborhood $\mathcal{N}$ in the space (Q_0, γ) (or (R_0, γ)) of the point characterizing A_* such that ∇T, where $\nabla = (\frac{\partial}{\partial Q_0}, \frac{\partial}{\partial \gamma})$ ($\nabla = (\frac{\partial}{\partial R_0}, \frac{\partial}{\partial \gamma})$), is everywhere continuous and nonvanishing.

This immediately implies that $\mathcal{N}$ is foliated smoothly by one-dimensional curves of constant period. As was the case for assumption H_1, this assumption is almost always satisfied in practice, but must be checked in specific cases. If assumption H_3 holds, then we may always coordinatize $\mathcal{N}$ with a local variable μ_2 and the wave period T itself, such that each constant period curve is coordinatized by μ_2 with T fixed. Of course, the coordinatization is not unique with respect to μ_2: all that we require is that the defining transformation

$$\mu_2 = \mu_2(\gamma, R_0)\,, \qquad T = T(\gamma, R_0)\,, \tag{4.10}$$

for centered waves, or

$$\mu_2 = \mu_2(\gamma, Q_0)\,, \qquad T = T(\gamma, Q_0)\,, \tag{4.11}$$

for uncentered waves, is nonsingular in $\mathcal{N}$. We therefore also define the assumption:

H_4: Assumption H_3 holds, and a particular nonsingular coordinatization (4.10) or (4.11) has been fixed over $\mathcal{N}$.

We can now state:

Proposition 4.3 *Given an uncentered or centered stationary linear phase wave A_* with parameters $T*$ and μ_2* that satisfies assumptions H_3 and H_4 and the necessary condition of Proposition 4.1, a sufficient condition for A_* to persist as a fully periodic stationary traveling wave with period T_* is that*

$$\left.\frac{\partial W^1(\mu_1,\mu_2,T)}{\partial \mu_2}\right|_{(\mu_1=0,\mu_2=\mu_2*,T=T_*)} \neq 0\,. \tag{4.12}$$

Proof: We prove the proposition for centered waves. The proof for uncentered waves is very similar and slightly simpler, and is therefore omitted. We work directly in the phase space (P, Q, R) of equations (3.6–3.8) with initial data

$$P_0 = 0\,, \quad Q_0 = 0\,, \quad R_0 = R_0(\mu_2, T)\,, \quad \gamma = \gamma(\mu_2, T)\,, \tag{4.13}$$

where the expressions for R_0 and γ are obtained by inverting transformation (4.10).

Equations (3.6–3.8), for $c = 0$, are

$$\partial_z P = -\frac{\epsilon}{1+\epsilon^2}\left(\alpha - h'(Q^2) + g'(Q^2)\right) Q^2, \tag{4.14}$$

$$\partial_z Q = R\,, \tag{4.15}$$

$$\partial_z R = \frac{P^2}{Q^3} + \alpha Q + h'(Q^2)Q + \frac{\epsilon^2}{1+\epsilon^2}\left(\alpha - h'(Q^2) + g'(Q^2)\right) Q\,. \tag{4.16}$$

In the unperturbed system ($\epsilon = 0$), for each fixed γ, each family of centered stationary linear phase solutions are periodic orbits that foliate an open set of the invariant plane $P = 0$ (in general, these open sets will be bounded by one or more separatrices separating families of centered waves and uncentered linear phase solutions). All of these solutions have Q profiles which are even functions in z about $Q_{\max}$. As can be inferred from the upcoming analysis, any persisting stationary linear phase solution will also have this property. Our goal then will be to show that a single, even, periodic orbit of the centered type persists in the vicinity of a centered solution that satisfies Proposition (4.3).

Clearly, all periodic orbits of centered type, which have even Q profiles about $Q_{\max}$ in either the perturbed or unperturbed equations, must pass through the R axis at two opposite points $(0, 0, R_0)$ and $(0, 0, -R_0)$, where R_0 is some real number, to avoid a singularity at $Q = 0$ in the term $\frac{P^2}{Q^3}$ in the second equation. It is therefore sufficient to look for periodic orbits of the perturbed system having initial data in $\mathcal{N}$ at points $(P_0, Q_0, R_0) = (0, 0, R_0(\mu_2, T))$, with $\alpha = \gamma(\mu_2, T)$, and which also intersect $(0, 0, -R_0(\mu_2, T))$.

We first prove that any persisting trajectory intersecting two opposite points on the R axis must also intersect the Q axis. This follows from the fact that system (4.14–4.16) is invariant under the symmetry

$$T_1 : P(z) \mapsto -P(-z)\,, \quad Q(z) \mapsto Q(-z)\,, \quad R(z) \mapsto -R(-z)\,. \tag{4.17}$$

Thus, if we have any trajectory starting at $(0, 0, R_0)$ and arriving at $(P^*, Q^*, 0)$ at some $z = z^*$, then by T_1 we have a 'time reversed' trajectory from $(0, 0, -R_0)$ to $(-P^*, Q^*, 0)$. To obtain a trajectory from $(0, 0, R_0)$ to $(0, 0, -R_0)$, we therefore need $P^* = 0$.

We now define a continuous segment of initial conditions $S_0 = (P_0, Q_0, R_0) = (0, 0, R_0(\mu_2, T_*))$, for all $\mu_2 \in \mathcal{N}$ in some open interval (μ_{2a}, μ_{2b}) containing μ_{2*}. Consider how the 'sheet' of trajectories generated by this segment intersects the P, Q plane. Let $(\Delta P(\epsilon, \mu_2), \Delta Q(\epsilon, \mu_2))$ describe the deviation of these intersection points from their unperturbed values. From simple estimates on equations (4.14–4.16) it follows that ΔQ will be $O(\epsilon^2)$, whereas ΔP will be $O(\epsilon)$. The necessary condition of Proposition 4.1, i.e. $W^1(0, \mu_{2*}, T_*) = 0$, is exactly the condition that $\Delta P(\mu_{2*}, T_*) = 0$ to $O(\epsilon)$. It follows easily that if the sufficient condition of Proposition (4.3) is met, i.e. that $W^1(0, \mu_2, T_*)$ has a simple zero on S_0 at μ_{2*}, the sheet will intersect the Q axis transversely to $O(\epsilon)$. There must therefore be a unique point of intersection, corresponding to some initial point $(0, 0, R_{**})$ which is $O(\epsilon)$ close to $(0, 0, R_0(\mu_{2*}, T_*))$. Figure 2 illustrates the idea.

We therefore have a trajectory from some point $(0, 0, R_{**})$ to $(0, 0, -R_{**})$, with even Q and R profiles and odd P profile (by T_1). But the system (4.14–4.16) is also invariant under the inversion symmetry

$$T_2 : P(z) \mapsto P(z)\,, \quad Q(z) \mapsto -Q(z)\,, \quad R(z) \mapsto -R(z)\,. \tag{4.18}$$

This allows us to immediately construct the other half of the periodic orbit by inversion.

We now show that the period of the persisting wave can be maintained to equal T_*. To this end, we define $\mathcal{T}(\mu_2, T, \epsilon)$ to be the 'time' it takes for the trajectory starting at $(0, 0, R_0(\mu_2, T))$ in the perturbed or unperturbed system to intersect the P, Q plane for the first time. Therefore, for example, the period of the persisting wave at some ϵ, assuming it has initial data (μ_{2**}, T_{**}), will have period $T = 4m\mathcal{T}(\mu_{2**}, T_{**}, \epsilon)$.

By the implicit function theorem, a sufficient condition for the persisting cen-

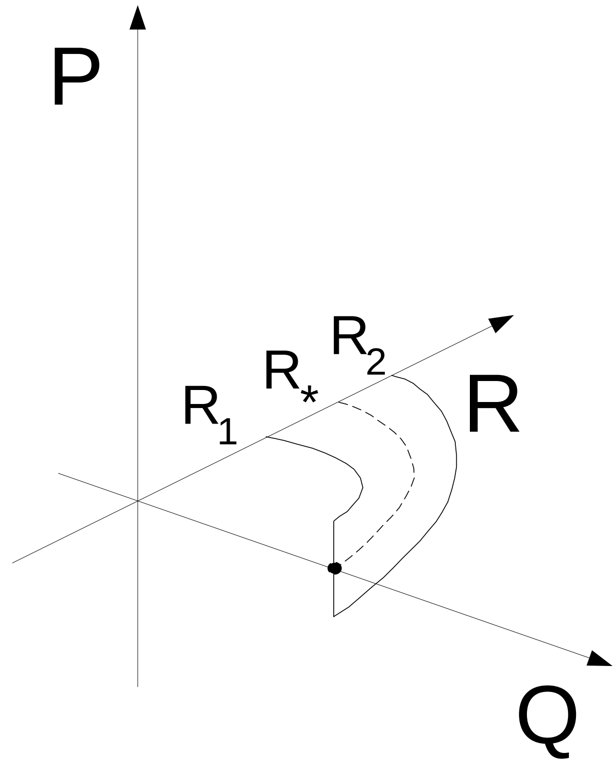

Fig. 2. Schematic illustration of how the sheet generated by S_0 transversely intersects the Q axis.

tered orbit to have initial data $(\mu_2(\epsilon), T(\epsilon))$, such that $\lim_{\epsilon\to 0}(\mu_2(\epsilon), T(\epsilon)) = (\mu_{2*}, T_*)$, and such that $\mathcal{T}(\mu_2(\epsilon), T(\epsilon), \epsilon) = T_*/(4m)$, is

$$\det \begin{pmatrix} W^1_{\mu_2} & W^1_T \\ \mathcal{T}_{\mu_2} & \mathcal{T}_T \end{pmatrix}\Bigg|_{(\mu_2=\mu_{2*},\epsilon=0)} = W^1_{\mu_2} \neq 0\,, \tag{4.19}$$

where we have used the fact that T is independent of μ_2. But this is the condition of Proposition (4.3).

To complete the proof, we need only to show that the phase of the persisting centered waves satisfy $S(T_*) = 0$. This follows from equation (2.16) (which gives the phase of perturbed centered waves, with μ replaced by the component P of the perturbed solution) with $c = 0$ and taking into account that P is odd and Q is even relative to $Q_{\max}$.

A Proof of Persistence Criteria for Uncentered Traveling Waves

In this section, we prove Proposition 3.3 for Persistence of Uncentered Traveling Wave Solutions, i.e. the criteria for persistence of uncentered traveling wave solutions. First, we derive necessary conditions and then the transversality conditions needed for sufficiency.

We begin by assuming that a candidate periodic uncentered traveling wave A_* has been identified, which has period T_*, speed c_*, modulation number m and phase winding number n. This wave corresponds to a unique solution of the unperturbed system (3.16–3.18) ($\epsilon = 0$) with initial data (P_0^*, I_0^*, γ^*).

From Proposition 3.2 we know that there exists a neighborhood $\mathcal{N}$ in the space (P_0, I_0, γ) of the initial data of A_* in which each point corresponds to a unique travelling wave with modulation number m and phase winding number n, such that these waves deform analytically to the A_* as their initial data approach that of A_*. We also know from the proposition that N can coordinatized by local variables μ_1, μ_2, and period T, which are related to the variables (P_0, I_0, γ) in this neighborhood by some nonsingular transformation. We now assume this transformation, call it F, to be fixed.

We now introduce a new variable $\tilde{c}(\epsilon)$, which is related to the speed c by

$$c = c_0(\mu_1, \mu_2, T) + \tilde{c}(\epsilon), \tag{A.1}$$

where $c_0(\mu_1, \mu_2, T)$ is the speed of the unperturbed traveling wave with modulation number m and phase winding number n with initial data (μ_1, μ_2, T).

Assuming this expression for c to be substituted into the system (3.16–3.18), the initial data of solutions of the system is now given by the variable quadraplet $(P_0, I_0, \gamma, \tilde{c})$, or equivalently by $(\mu_1, \mu_2, T, \tilde{c})$, where (μ_1, μ_2, T) are related to (P_0, I_0, γ) by the transformation F. We henceforth will use these quadraplets interchangeably. We also henceforth consider the variables in either quadraplet to be independent. This is justified further below. Note that the variable $\tilde{c}$ must be included because it plays a role in the perturbed system. It is also important to keep in mind that the solution of the *perturbed* system with initial data (μ_1, μ_2, T, c) does not necessarily have period T/m or is even necessarily periodic.

We next define a Poincaré section to be all points in the phase space (P, I, Θ) of system (3.16–3.18) such that

$$(\Theta - \Theta_*) = 0 \qquad (\mathrm{mod}\ 2\pi)\,, \tag{A.2}$$

where Θ_* is some arbitrary real number.

We want to find conditions as functions of initial data that when satisfied at $(\mu_{1*}, \mu_{2*}, T_*, \tilde{c} = 0)$ ensure that there is an periodic orbit of the perturbed system (3.16–3.18) with period T_*/m for all ϵ less than some constant

$\epsilon_0 > 0$, such that this orbit always generates an uncentered traveling wave with period T_* and phase winding number n. To cast this problem as a persistence problem, we require the persisting periodic orbit to have initial data $(P_0(\epsilon), I_0(\epsilon), \gamma(\epsilon), \tilde{c}(\epsilon))$, strictly on the Poincaré section (i.e. $\Theta_0(\epsilon) = \Theta_*$), such that

$$\lim_{\epsilon \to 0}(P_0(\epsilon), I_0(\epsilon), \gamma(\epsilon), \tilde{c}(\epsilon)) = (P_*, I_*, \gamma_*, 0)\,. \tag{A.3}$$

We now define

$$\begin{aligned} P &\equiv \mathcal{P}(z, P_0, I_0, \gamma, \tilde{c}, \epsilon)\,, \\ I &\equiv \mathcal{I}(z, P_0, I_0, \gamma, \tilde{c}, \epsilon)\,, \\ \Theta &\equiv \vartheta(z, P_0, I_0, \gamma, \tilde{c}, \epsilon)\,, \end{aligned} \tag{A.4}$$

to be the solution of the perturbed problem (3.16–3.18) at 'time' z with initial data $(P_0, I_0, \gamma, \tilde{c})$ on the Poincaré section, and we define

$$\mathcal{T} = \mathcal{T}(P_0, I_0, \gamma, \tilde{c}, \epsilon)\,, \tag{A.5}$$

to be the time at which

$$\vartheta(\mathcal{T}, P_0, I_0, \gamma, \tilde{c}, \epsilon) - \Theta_* = 2\pi m\,. \tag{A.6}$$

Therefore, $\mathcal{T}$ is the time at which the solution hits the Poincaré section after Θ makes m oscillations. The fact that the perturbation does not introduce any singularities is enough to ensure the existence of $\mathcal{T}$ for initial data in $\mathcal{N}$ and sufficiently small ϵ.

Thus, in part, we seek initial data such that

$$\mathcal{P}(\mathcal{T}(P_0, I_0, \gamma, \tilde{c}, \epsilon), P_0, I_0, \gamma, \tilde{c}, \epsilon) - P_0 = 0\,, \tag{A.7}$$

$$\mathcal{I}(\mathcal{T}(P_0, I_0, \gamma, \tilde{c}), P_0, I_0, \gamma, \tilde{c}, \epsilon) - I_0 = 0\,. \tag{A.8}$$

Note that the definition of $\mathcal{T}$ implies that any solution of these equations necessarily occurs on the Poincaré section. But we know that at $\epsilon = 0$ *any* point (P_0, I_0, γ) in $\mathcal{N}$ and any c solves this system. Thus these equations as they stand are singular at $\epsilon = 0$ and do not yield the conditions we seek.

To find some nonsingular functions we divide these equations by ϵ and look for a continuous family $(P_0(\epsilon), I_0(\epsilon), \gamma(\epsilon)), \tilde{c}(\epsilon))$ that satisfies (A.3) and solves the system

$$\frac{\mathcal{P}(\mathcal{T}(P_0,I_0,\gamma,\tilde{c},\epsilon),P_0,I_0,\gamma,\tilde{c},\epsilon)-P_0}{\epsilon}=0\,, \tag{A.9}$$

$$\frac{\mathcal{I}(\mathcal{T}(P_0,I_0,\gamma,\tilde{c},\epsilon),P_0,I_0,\gamma,\tilde{c},\epsilon)-I_0}{\epsilon}=0\,. \tag{A.10}$$

Note that for $\epsilon > 0$ these equations are equivalent to (A.8).

For any such family $(P_0(\epsilon), I_0(\epsilon), \gamma(\epsilon)), \tilde{c}(\epsilon))$ that satisfies (A.3), regardless of whether or not equations (A.9) and (A.10) are satisfied, we can define the limit of these equations as $\epsilon \to 0$, which are the Melnikov functions for the persistence of the orbit at $(P_*, I_*, \gamma_*, 0)$. Using the fact that P and I are constant at $\epsilon = 0$ and also that for any smooth functions $g(\cdot,\epsilon)$ and $h(\epsilon)$ we have the identity

$$\lim_{\epsilon\to 0}\frac{g(h(\epsilon),\epsilon)-g(h(\epsilon),0)}{\epsilon}=\partial_\epsilon g(h(0),\epsilon)\Big|_{\epsilon=0}\,,$$

we have

$$\begin{aligned}
&M^1(P_*,I_*,\gamma_*,0) \\
&\equiv \lim_{\epsilon\to 0}\frac{\mathcal{P}(\mathcal{T}(P_0,I_0,\gamma,\tilde{c},\epsilon),P_0,I_0,\gamma,\tilde{c},\epsilon)-P_0}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\mathcal{P}(\mathcal{T}(P_0,I_0,\gamma,\tilde{c},\epsilon),P_0,I_0,\gamma,\tilde{c},\epsilon)-\mathcal{P}(\mathcal{T}(P_0,I_0,\gamma,\tilde{c},\epsilon),P_0,I_0,\gamma,\tilde{c},0)}{\epsilon} \\
&= \partial_\epsilon \mathcal{P}(\mathcal{T}(P_*,I_*,\gamma_*,0,0),P_*,I_*,\gamma_*,0,\epsilon)\Big|_{\epsilon=0} \\
&= \int_0^{T_*} f_1(P_*,I_*,\Omega_*+\Omega(P_*,I_*,\gamma_*)z,\gamma_*,c_*,0)\,dz\,,
\end{aligned} \tag{A.11}$$

and

$$\begin{aligned}
&M^2(P_*,I_*,\gamma_*,c_*) \\
&\equiv \lim_{\epsilon\to 0}\frac{\mathcal{I}(\mathcal{T}(P_0,I_0,\gamma,\tilde{c},\epsilon),P_0,I_0,\gamma,\tilde{c},\epsilon)-I_0}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\mathcal{I}(\mathcal{T}(P_0,I_0,\gamma,\tilde{c},\epsilon),P_0,I_0,\gamma,\tilde{c},\epsilon)-\mathcal{I}(\mathcal{T}(P_0,I_0,\gamma,\tilde{c},\epsilon),P_0,I_0,\gamma,\tilde{c},0)}{\epsilon} \\
&= \partial_\epsilon \mathcal{I}(\mathcal{T}(P_*,I_*,\gamma_*,0,0),P_*,I_*,\gamma_*,0,\epsilon)\Big|_{\epsilon=0} \\
&= \int_0^{T_*} f_2(P_*,I_*,\Omega_*+\Omega(P_*,I_*,\gamma_*)z,\gamma_*,0,0)\,dz\,,
\end{aligned} \tag{A.12}$$

Using (**??**), we can recast these functions as functions of $(\mu_1, \mu_2, T, \tilde{c})$, i.e. $M^{(1,2)}(\mu_{1*}, \mu_{2*}, T_*, c_*)$.

Clearly, a necessary condition for a family $(P_0(\epsilon), I_0(\epsilon), \gamma(\epsilon)), c(\epsilon))$ to satisfy $(A.8)$ for every ϵ in some interval $0 \le \epsilon < \epsilon_0$ is that

$$M^1(\mu_{1*}, \mu_{2*}, T_*, c_*) = M^2(\mu_{1*}, \mu_{2*}, T_*, c_*) = 0\,. \tag{A.13}$$

If we substitute now the expressions (3.19) and (3.20) for f_1 and f_2, and use the following relations

$$\frac{\partial I}{\partial R} = \frac{T}{2\pi} R\,, \tag{A.14}$$

$$\frac{\partial I}{\partial P} = \frac{T}{2\pi}\frac{P}{Q^2}\,, \tag{A.15}$$

we obtain (dropping the asterix on the independent variables)

$$M^1(\mu_1, \mu_2, T, c) = \int_0^T \left(cP + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2)\right) Q^2 \right) dz\,,$$

$$M^2(\mu_1, \mu_2, T, c) = \int_0^T \left(c\frac{P^2}{Q^2} + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2)\right) P + cR^2 \right) dz\,,$$

where $\alpha = \gamma(\mu_1, \mu_2, T) - c^2/4$ and where P is determined from (**??**) and Q and R are functions determined from the composition of (3.15) with (**??**). It is an easy exercise to show that these Melnikov functions are linear combinations of, and are therefore equivalent to, the Melnikov functions (1.4) and (1.5) found using the averaging method in Section 3.

We now determine sufficient conditions for persistence. First, we define a symbol for the phase S generated by the perturbed solution at $z = T$,

$$\mathcal{S}(\mu_1, \mu_2, T, c, \epsilon) \equiv \int_0^T \left(\frac{c}{2} + \frac{\mathcal{P}(z)}{\mathcal{Q}(z)^2} \right) dz\,, \tag{A.16}$$

where $\mathcal{P}(z)$ is the same as defined in (A.4) and $\mathcal{Q}(z)$ is obtained by composition from (3.15) and (A.4).

We wish to determine conditions that ensure system (A.9–A.10) has a solution family $(\mu_1(\epsilon), \mu_2(\epsilon), T(\epsilon), c(\epsilon))$ (expressed now in the local variables) satisfying (A.3) and also such that

$$\begin{aligned} &\mathcal{S}(\mu_1(\epsilon), \mu_2(\epsilon), T(\epsilon), c(\epsilon)) = 2\pi n\,, \\ &\mathcal{T}(\mu_1(\epsilon), \mu_2(\epsilon), T(\epsilon), c(\epsilon)) = T_*\,, \end{aligned} \tag{A.17}$$

where we have also expressed $\mathcal{T}$ in the local variables.

By the implicit function theorem, a sufficient condition for such a family to exist is that

$$\det \begin{pmatrix} M^1_{\mu_1} & M^1_{\mu_2} & M^1_T & M^1_c \\ M^2_{\mu_1} & M^2_{\mu_2} & M^2_T & M^2_c \\ \mathcal{T}_{\mu_1} & \mathcal{T}_{\mu_2} & \mathcal{T}_T & \mathcal{T}_c \\ \mathcal{S}_{\mu_1} & \mathcal{S}_{\mu_2} & \mathcal{S}_T & \mathcal{S}_c \end{pmatrix} \Bigg|_{(\mu_1=\mu_{1*},\mu_2=\mu_{2*},T=T_*,c=c_*,\epsilon=0)} \neq 0\,. \tag{A.18}$$

However, at $\epsilon = 0$ we have $\mathcal{T} = T$ (by the definition of T), so that $\mathcal{T}_{\mu_1} = \mathcal{T}_{\mu_2} = \mathcal{T}_c = 0$ and $\mathcal{T}_T = 1$. Also, we have $\mathcal{S}_c = 1/2$, because $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ are independent of c at $\epsilon = 0$. Therefore, the sufficient condition reduces to

$$\det \begin{pmatrix} M^1_{\mu_1} & M^1_{\mu_2} & M^1_c \\ M^2_{\mu_1} & M^2_{\mu_2} & M^2_c \\ S_{\mu_1} & S_{\mu_2} & \frac{1}{2} \end{pmatrix} \Bigg|_{(\mu_{1*},\mu_{2*},T_*,c_*)} \neq 0\,, \tag{A.19}$$

which completes the proof of proposition (3.3).

As mentioned in Section 3, we wish to look for persisting waves of period T by analyzing only the Melnikov functions on period T waves of the unperturbed system. To this end we set $c = c(\mu_1, \mu_2, T)$, which restricts the speed so that the unperturbed wave is periodic, and defined 'restricted' Melnikov functions

$$W^{(1,2)}(\mu_1, \mu_2, T) \equiv M^{(1,2)}(\mu_1, \mu_2, T, c(\mu_1, \mu_2, T))\,. \tag{A.20}$$

On the restricted set of speeds the vanishing of these functions is clearly identical to the necessary condition given above.

We may still determine conditions that ensure system (A.9–A.10) has a solution family $(\mu_1(\epsilon), \mu_2(\epsilon), T(\epsilon))$ (with $c(\epsilon) = c(\mu_1(\epsilon), \mu_2(\epsilon), T(\epsilon))$) satisfying (A.3) such that either

$$\mathcal{S}(\mu_1(\epsilon), \mu_2(\epsilon), T(\epsilon), c(\epsilon)) = 2\pi n\,, \tag{A.21}$$

or

$$\mathcal{T}(\mu_1(\epsilon), \mu_2(\epsilon), T(\epsilon), c(\epsilon)) = T_*\,, \tag{A.22}$$

but not both.

By the implicit function theorem, a sufficient condition to satisfy (A.9–A.10) and (A.22) is

$$\det \begin{pmatrix} W^1_{\mu_1} & W^1_{\mu_2} & W^1_T \\ W^2_{\mu_1} & W^2_{\mu_2} & W^2_T \\ \mathcal{T}_{\mu_1} & \mathcal{T}_{\mu_2} & \mathcal{T}_T \end{pmatrix} \Bigg|_{(\mu_1=\mu_{1*},\mu_2=\mu_{2*},T=T_*,c=c_*,\epsilon=0)} \neq 0\,. \tag{A.23}$$

Once again using the fact that $\mathcal{T}_{\mu_1} = \mathcal{T}_{\mu_2} = 0$ and $\mathcal{T}_T = 1$, we obtain

$$\frac{\partial(W^1, W^2)}{\partial(\mu_1, \mu_2)}\bigg|_{(\mu_{1*},\mu_{2*},T_*)} \neq 0\,, \tag{A.24}$$

which completes the proof of proposition 3.4. This condition is sufficient for a period T_* to persist in system (A.9–A.10), but simply gives us no information on whether $\mathcal{S}$ can be made to remain equal to $2\pi n$.

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