Complex Ginzburg-Landau Equations as Perturbations of Nonlinear Schrödinger Equations: A Case Study

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Abstract

As a case study of theory developed in two previous papers, the persistence of spatially periodic solutions of the focusing nonlinear cubic Schrödinger equation under cubic complex Ginzburg-Landau perturbation is studied in detail. The persistence of nontrivial traveling waves is related to bifurcations involving traveling waves and rotating waves.

1 Introduction

This paper is the third in a series of three papers (see also [1,2]) and also some other work by some of us [4–7] in a program concerned with the long-

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time dynamics of a class of infinite dimensional Hamiltonian systems when the symmetries of these systems are broken by a dissipative perturbation.

Specifically in this series, we study the persistence of solutions of the generalized nonlinear Schrödinger (GNLS) equation

$$\partial_t A = i\partial_{xx} A - ih'(|A|^2)A \,, \tag{1}$$

when the GNLS equation is perturbed to a generalized complex Ginzburg-Landau (GCGL) equation

$$\partial_t A = (i+\epsilon)\partial_{xx} A - ih'(|A|^2)A - \epsilon\, g'(|A|^2)A \,. \tag{2}$$

Here $\epsilon > 0$, while $h = h(\xi)$ and $g = g(\xi)$ are real analytic functions over $[0, \infty)$. In the first paper of this series [1], necessary conditions for the persistence of general spatially and temporally quasiperiodic and spatially periodic homoclinic solutions were derived directly from the partial differential equations using an averaging method. In the second paper [2], the necessary and sufficient conditions for persistence of traveling waves were developed in much greater detail using the ordinary differential equations governing traveling waves.

In this paper we present a case study of the conditions for persistence of traveling waves obtained in [1,2]. We take as our example the ubiquitous focusing cubic Ginzburg-Landau (fcCGL) equation,

$$\partial_t A = \epsilon r A + (\epsilon + i)\partial_{xx} A - 2(\epsilon q - i)|A|^2 A \,. \tag{3}$$

For this case, the functions h and g have the power law forms

$$h(\xi) = -\xi^2 \,, \qquad g(\xi) = -r\xi + q\xi^2 \,. \tag{4}$$

At $\epsilon = 0$, the fcCGL reduces to the focusing (and integrable [8,9]) cubic nonlinear Schrödinger (fcNLS) equation

$$\partial_t A = i\partial_{xx} A + 2i|A|^2 A \,. \tag{5}$$

By a traveling wave solution of these equations, we mean any solution of the form

$$A(x,t) = Q(x-ct)e^{-i\alpha t + iS(x-ct)} \,, \tag{6}$$

where α and c are real constants and S and Q are periodic real-valued functions with some period T. The exact periodic traveling wave solutions of (5) are given in the next section.

The present paper draws heavily on the results of [2]. We therefore recommend that the reader become familiar with [2] before attempting to read the present paper.

2 Explicit Expressions for fcNLS Waves

In section 2 of [2] the traveling wave solutions for GNLS equations of the form (1) are classified into well defined types called even/odd, centered, waves and uncentered, linear/nonlinear phase, waves, and a precise definition of a "family" of traveling waves was given. We now briefly review the principle features of this classification.

From [2], the amplitude profiles $Q(z)$ of GNLS traveling waves (where $z = x - ct$) correspond to solution trajectories $Q(z)$ in the reflection symmetric potential

$$V(Q;\mu,\gamma) = \frac{\mu^2}{Q^2} + \gamma Q^2 - h(Q^2)\,, \tag{7}$$

where μ is an integration constant related to the traveling wave phase profile $S(z)$ by

$$S(z) = \int_0^z \left(\frac{c}{2} + \frac{\mu}{Q^2(z')} \right) dz'\,. \tag{8}$$

and γ is a lumped parameter related to the wave speed c and frequency α (as defined by (6)) by

$$\gamma \equiv \frac{c^2}{4} + \alpha\,. \tag{9}$$

According to the classification given in [2], waves called **centered** are those having amplitude profiles $Q(z)$ which are symmetric about $Q = 0$, i.e, $Q(z)$ is invariant under a sign change and a translation. Waves called **uncentered** are those associated with profiles asymmetric about $Q = 0$ which due to the reflection symmetry of $V(Q;\mu,\gamma)$ implies that Q is everywhere of one sign (we take $Q > 0$ without loss of generality).

Centered waves are further classified as **even/odd** if their amplitude profiles execute an integral number m of full/half orbits per period T of the wave. We call m the **modulation number**.

Finally, waves are classified as being **linear/nonlinear phase** waves, depending on whether or not $\mu = 0$. By (8), the phase $S(z)$ is a linear function in z if $\mu = 0$, and is generally nonlinear for $\mu \neq 0$. Uncentered waves can be either linear or nonlinear phase waves, whereas centered waves are always of linear phase type. For uncentered waves, the phase $S(z)$ always changes (winds) by an amount $2\pi n$ per wave period, where n is an integer. For centered waves, the phase $S(z)$ changes by an amount $n\pi$, where n is even/odd if the wave is of even/odd type (so that for a centered wave both m and n are both even or both odd). For all waves we call n the **phase number** instead of the more usual term "winding number" because for centered waves the winding number is undefined for the reason that the amplitude profile Q alway possesses zeros.

A **family** of waves was defined in Definition 2.0.1 of [2] to be a set of waves of a particular type, that is, even/odd centered or uncentered type, and a particular period T, modulation number m, and phase number n. Each family is also to be associated with at most one particular subwell of the potential, where the subwells are partitioned from one another by heteroclinic and homoclinic orbits connecting critical points of the potential. Thus if there is more than one potential well, a subwell index will also be needed. To allow for situations in which there are more than one orbit in the same subwell with the same modulation number m for a given wave period T and values of μ and γ, we simply *define* the family to be the set of all such orbits. It will also be useful to define the notion of a **subfamily**. This is defined to be the largest subset of a family for which μ has one particular sign or is zero.

Under this definition of a family, it was noted in [2] that for a given instance of the GNLS equation, uncentered families are (generically) internally parameterized by only two continuous parameters μ_1 and μ_2 (for example, one can often choose $\mu_1 = \mu$ and $\mu_2 = \gamma$ to parameterize at least some subregion of a family), whereas families of even/odd centered waves and subfamilies of uncentered linear phase waves are (generically) internally parameterized by only a single continuous parameter, say μ_2, due to the restriction $\mu = 0$ ($\mu_1 = 0$).

We now discuss the specific traveling wave solutions of the fcNLS equation (5). From (7) and (4) we find that the potential governing the traveling wave amplitude profiles of the fcNLS equation (5) is

$$V(Q;\mu,\gamma) = \frac{\mu^2}{Q^2} + \gamma Q^2 + Q^4 \,. \tag{10}$$

For $\mu \neq 0$ and all values of γ, the potential has two local minima located symmetrically around a singularity at the origin, forming two asymmetric potential wells. Due to their asymmetry these wells only give rise to uncentered nonlinear phase waves. Because of the reflection symmetry of $V(Q;\mu,\gamma)$ we restrict ourselves to $Q > 0$ for uncentered waves without loss of generality, and so for each choice of T, n, and m we obtain two subfamilies of uncentered waves, one set with $\mu > 0$ and one set with $\mu < 0$. The relationship of these subfamilies to a family is discussed after we next discuss the $\mu = 0$ case. Note that the amplitude profiles of these two subfamilies are in a one-to-one correspondence because μ is squared in $V(Q; \mu, \gamma)$, but that waves with corresponding amplitude profiles generally have different phase profiles, speeds, and frequencies.

For $\mu = 0$, there is no singularity and there are two basic shapes of the potential, depending on the sign of γ. For $\gamma < 0$, the potential has two local minima at negative energies located symmetrically about a local maxima at zero energy at the origin. In this case the two potential wells associated with negative energies give rise to uncentered linear phase waves and at positive energies the potential gives rise to even centered waves and odd centered waves. For $\gamma \geq 0$, there is only a single minima at the origin, and only even centered waves and odd centered waves exist, all with positive energy. From these two configurations taken together (both signs of γ), for each choice of T, n and m we obtain one subfamily of uncentered linear phase waves (restricting to the profiles with $Q > 0$ only), and one family (not subfamily) apiece of even centered waves and odd centered waves (both families having members with both signs of γ).

Together, the three subfamilies of uncentered waves obtained for $\mu > 0$, $\mu = 0$ and $\mu < 0$, respectively, for each choice of T, n, m, form a single family of uncentered waves.

We now give the exact expressions for these waves. Up to the arbitrary z-translation and phase, the even/odd centered waves of period T are given by

$$Q(z) = \lambda\kappa\,\mathrm{cn}(\lambda z, \kappa)\,, \tag{11}$$

$$S(z) = \frac{c}{2} z\,, \tag{12}$$

$$c = \frac{2\pi n}{T}\,, \tag{13}$$

$$\alpha = -2\lambda^2\kappa^2 + \lambda^2 - c^2/4\,, \tag{14}$$

where the parameter λ is defined by

$$\lambda \equiv \frac{2m\,\mathrm{K}(\kappa)}{T}\,. \tag{15}$$

The function $\mathrm{K}(\kappa)$ is the complete elliptic integral of the first kind. The function $\mathrm{cn}(x, \kappa)$ is the Jacobi cnoidal elliptic function with modulus κ, $0 \leq \kappa < 1$, and has m half periods in the interval $[0, T]$. The integer n can be either sign.

To obtain the even centered waves, m and n must both be even, but otherwise independent. For the odd waves, m and n must both be odd, but otherwise independent. The amplitude profiles $Q(z)$ for an even and an odd centered wave are plotted in Figure 1(a-d) for two values of κ. Note that for small κ these waves are close to sinusoids (to which they deform as $\kappa \to 0$.

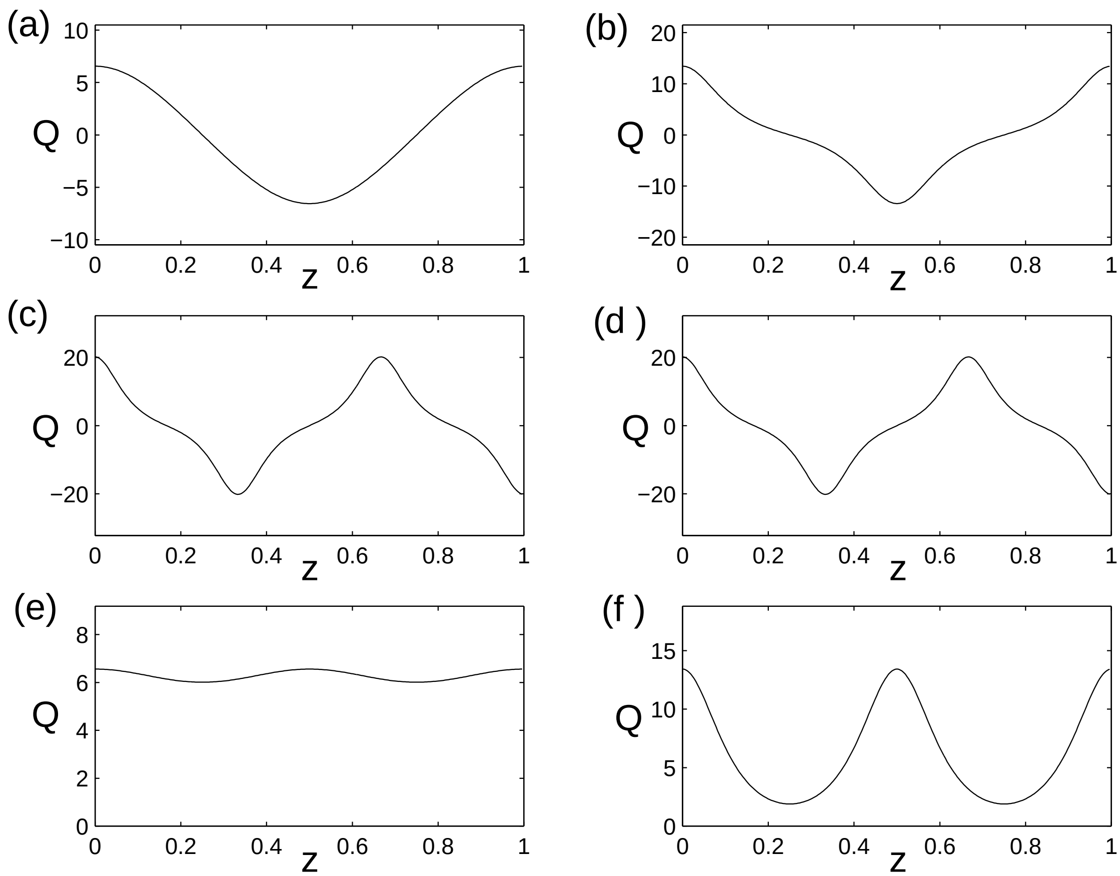

Fig. 1. Amplitude profiles for (a) even uncentered wave with $m = 2$, $\kappa = .4$, (b) even uncentered wave with $m = 2$, $\kappa = .99$ (c) odd uncentered wave with $m = 3$, $\kappa = .4$, (d) odd uncentered wave with $m = 3$, $\kappa = .99$, (e) uncentered linear phase wave with $m = 2$, $\kappa = .4$, (f) uncentered linear phase wave with $m = 2$, $\kappa = .99$.

Up to the arbitrary z-translation and phase, the periodic uncentered waves of period T are found to be

$$Q(z) = \lambda\sqrt{\left(\mathrm{dn}^2(\lambda z, \kappa) - 1 + \delta^2\right)}\,, \tag{16}$$

$$S(z) = \int_0^z \left(\frac{c}{2} + \frac{\mu}{Q^2(z')}\right) dz'\,, \tag{17}$$

$$c = \frac{4\pi n}{T} - \frac{4m\mu}{T}\Pi\left(\kappa, -\frac{\kappa^2}{\delta^2}\right)\sqrt{(1-\delta^2)\left(1-\frac{\kappa^2}{\delta^2}\right)}\,, \tag{18}$$

$$\alpha = -\frac{c^2}{4} + \lambda^2(1 - 3\delta^2 + \kappa^2)\,, \tag{19}$$

$$\mu^2 = \lambda^6\delta^2(1-\delta^2)(\delta^2 - \kappa^2)\,. \tag{20}$$

with κ and λ defined the same as above for centered waves and where $\Pi(\kappa, b)$ is the complete elliptic integral of the third kind. The function $\mathrm{dn}(z, \kappa)$ is the Jacobi dnoidal elliptic function with modulus κ, and has m full periods in the interval $[0, T]$. The parameters δ and κ are constrained to lie within the 'simplex' defined by the inequalities

$$0 < \kappa < \delta \leq 1\,. \tag{21}$$

The function $Q(z)$ has a maximum value of $\lambda\delta$ and a minimum value of $\lambda\sqrt{\delta^2 - \kappa^2}$. As an example, the amplitude profile of an uncentered linear phase wave (see Remark 2.0.2) is plotted in Figure 1. Note that for small κ, the amplitude is basically a sinusoidal perturbation on a constant background. This limit is explored further in Remark 2.0.1.

We obtain particular families of waves from the expressions above by fixing T, n, and m. Note that because the period T and the phase and modulation numbers n and m are independent of δ and κ, the parameters δ and κ, along with the sign of μ, form "internal" parameters of the traveling wave families. Specifically, when the sign of μ is fixed and μ is nonzero, the parameters δ and κ restricted to the simplex (21) completely parameterize one of the two subfamilies of nonlinear phase uncentered waves possessed by each family of uncentered waves. At $\mu = 0$, which for the uncentered waves also implies $\delta = 1$, the parameter κ $(0 < \kappa < 1)$ completely parameterizes each subfamily of linear phase uncentered waves and each family of centered waves.

Remark 2.0.1 *The fcNLS equation possesses a set of simple planewave solutions of period T often called "rotating waves" given by*

$$A^{(n)}(x,t) = a e^{i(\frac{2\pi n}{T}x - [-2a^2 + \frac{4\pi^2 n^2}{T^2}]t + \sigma)}\,, \tag{22}$$

where a and σ are real positive constants and n is the phase winding number. When the modulus κ in the amplitude (16) goes to zero, the family of uncentered waves with period T, modulation number m and phase number n deforms smoothly to the subset of these rotating waves with

$$a = \frac{m\pi\delta}{T}\,, \tag{23}$$

where $0 \leq \delta \leq 1$ from (21). Thus, each set of rotating waves with a specific T and winding number n is a degenerate limit of infinitely many families of uncentered traveling waves. It was shown at the end of section 2 of [2] that the subset of limiting rotating waves to which the family of uncentered waves with modulation number m deform are neutrally stable with respect to "sideband" perturbations with wavenumber $k_m = 2\pi m/T$, i.e., a sideband perturbation in this direction generates the nearby uncentered waves of the family. The remaining rotating waves are modulationally unstable with respect to sideband perturbations with wavenumber k_m, and instead of being limiting members of this uncentered family, these rotating waves are the homoclinic points of homoclinic orbits [10].

Remark 2.0.2 *Another important special case of the uncentered waves that we will treat in this paper is obtained by taking $\delta \to 1$. In this limit $\mu \to 0$, and we obtain the uncentered linear phase waves*

$$A(x,t) = \lambda \,\mathrm{dn}(\lambda(x-ct),\kappa) e^{i\left(\frac{2\pi n}{T}x - [\left(\frac{2\pi n}{T}\right)^2 - \lambda^2(2-\kappa^2)]t\right)} . \tag{24}$$

By choosing a specific T, m and n we obtain a particular subfamily of uncentered linear phase waves at $\mu = 0$. We see that this subfamily is smoothly embedded between the two subfamilies uncentered nonlinear phase waves with $\mu \neq 0$.

The various limits of the uncentered waves discussed above are illustrated in Figure 2.

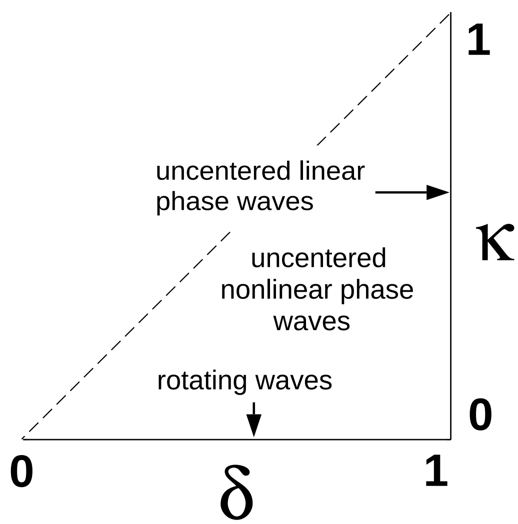

Fig. 2. Limits of the uncentered waves in the κ, δ simplex.

Remark 2.0.3 *For each family of uncentered waves that deform to a subset of rotating waves as in Remark (2.0.1), both of the subfamilies of nonlinear phase waves with $\mu \neq 0$ deform to the same set of rotating waves. It is there-*

fore tempting to view the set of rotating waves as a kind of subfamily which is smoothly embedded in a family of uncentered waves, analogous to the subfamilies at $\mu = 0$ described in Remark (2.0.2). Note, however, that the values of μ and the speed c of the traveling waves suffer a discontinuous jump at the rotating waves (these values are double valued at the rotating waves). This means that certain sufficient persistence criteria which we will use later on involving transversality conditions cannot be applied at the rotating waves. Fortuneately, the persistence of rotating waves can be calculated explicitly.

3 Parameterization of Traveling Wave Families for persistence analysis

Having obtained explicit expressions for the fcNLS traveling waves, we now take a preliminary step toward discussing their persistence under the fcCGL perturbation. To analyze persistence, which is carried out in the further sections of this paper, we will appeal to the propositions giving persistence criteria in sections 3 and 4 of [2]. These propositions were stated in terms of special internal parameters of the traveling wave families which satisfy some special preconditions. In this section we choose these parameters and demonstrate the reason for their use.

In section 3 of [2] a system of ordinary differential equations in action-angle variables (system (49)-(51) of [2]) governing the profiles of uncentered GCGL traveling waves was derived from which the persistence criteria for uncentered waves was obtained. This system was written in action-angle variables (P, I, Θ) (two actions, one angle), and when $\epsilon = 0$ the system depends only on the lumped parameter γ, the parameter μ being the action P at $\epsilon = 0$. So, up to a trivial phase, an amplitude profile Q of an fcNLS uncentered wave is uniquely determined by giving the initial data (P_0, I_0, γ). This space of initial data, and analogous spaces of initial data for linear phase waves, plays an important role in the following.

Following [2], we assume that a particular wave has been choosen to be tested for persistence, which implies that a particular choice of modulation number m has also been made. We then seek to reparameterize the space of initial data (P_0, I_0, γ), at least locally in some neighborhood $\mathcal{N}$ containing the point corresponding to the candidate wave, by the wave period T of the unperturbed waves corresponding to each point (which are determined by (P_0, I_0, γ, m)) and some new parameters μ_1 and μ_2. The new parameters μ_1 and μ_2 are chosen specifically such that they parameterize the surfaces of constant period T in $\mathcal{N}$, if such surfaces exist. We will label any such surface Σ_T. Because μ_1 and μ_2 parameterize surfaces of constant period, these parameters are (abstract) internal parameters of families of fcNLS traveling waves. The reparametization exists, at least locally, if the technical assumption called H_1 in [2] is met, which is that the gradient of T in the space of initial data is defined and does not vanish in $\mathcal{N}$. It is possible that this condition may not be met at points in (P_0, I_0, γ) for some GNLS equations, for example, if the potential well is harmonic. At such points, the results of [2] cannot be applied (this is never the case for the fcNLS equation). The reason for using such variables is to avoid singularities that result if one attempts to work only with the variables P_0, I_0, and γ. We will explicitly demonstrate such singularities for the fcNLS equation below.

To obtain persistence criteria for linear phase waves, another system was derived in [2], system (77)-(79) in section 4 , and the corresponding spaces of initial data used were (γ, Q_0) for uncentered linear phase waves, and (γ, R_0) for centered waves. In this case, a reparameterization in terms of the variables μ_2 and T was defined (we may think of μ_1 also being defined with $\mu_1 \equiv \mu = 0$), so that μ_2 internally parameterizes the linear phase (sub)families. This reparameterization exists at least locally if the technical assumption called H_3, analogous to H_1 in section 4 of [2] is met.

We now choose definitions for μ_1 and μ_2 for the fcNLS equation in terms of the "natural" internal parameters δ, κ. For uncentered nonlinear phase waves, we choose

$$\mu_1 \equiv \delta\,, \quad \mu_2 \equiv \kappa: \text{ (uncentered waves)}, \tag{25}$$

where the subfamilies at $\mu > 0$ and $\mu < 0$ are to be handled separately. Likewise, for linear phase waves, we choose

$$\mu_2 \equiv \kappa: \text{ (linear phase waves)}. \tag{26}$$

We have verified numerically that these two choices satisfy assumptions H_1 and H_3 of [2] in the δ, κ simplex (21), by numerically computing the jacobian of the transformation from (δ, κ, T) (which is now also (μ_1, μ_2, T)) to (P_0, I_0, γ). This transformation is given explicitly by

$$P_0 = \mu = \pm \frac{8m^3\,\mathrm{K}^3(\kappa)}{T^3}\delta\sqrt{(1-\delta^2)(\delta^2-\kappa^2)}\,, \tag{27}$$

$$I_0 = \frac{8m^4\,\mathrm{K}^3(\kappa)}{T^3\pi}\int_0^{\mathrm{K}(\kappa)} \frac{(\,\mathrm{dn}(y,\kappa)\frac{\partial\,\mathrm{dn}(y,\kappa)}{\partial y})^2}{\mathrm{dn}^2(y,\kappa) - 1 + \delta^2}\,dy\,, \tag{28}$$

$$\gamma = \frac{4m^2\,\mathrm{K}^2(\kappa)}{T^2}\left(-3\delta^2 + (1+\kappa^2)\right)\,. \tag{29}$$

Note that the singularity or nonsingularity of this transformation does not depend on m. The condition that the gradient of T in the spaces of initial data is continuous and nonvanishing, which is needed for assumptions H_1 and H_3, follows immediately from the simple power law dependencies of these functions on T.

We now describe the structure of the surfaces Σ_T which the transformation defines implicitly, to clearly demonstrate the reason for using the internal variables μ_1 and μ_2. This also sheds light on what can be expected in cases other than the fcNLS case.

The structure of the surfaces Σ_T for the fcNLS equation is illustrated in Figure 3, which shows a finite part of the surface at $m = T = 1$ (the part for $\kappa < .999$. As δ and κ together approach 1, I_0 and γ diverge, so that it is impossible to show the whole surface). As the labeling of the figure indicates, this surface may be thought of as two δ, κ simplexes embedded in the space (P_0, I_0, γ) and glued smoothly together along the edge at $\delta = 1$, $0 < \kappa < 1$, and touching also at the point $\delta = \kappa = 0$. The subfamily of linear phase uncentered waves at $\mu = 0$ lie along the curve where the simplixes are glued. The rotating waves exist on the closed curve formed by the intersection of Σ_T with the P_0, γ plane. Note that pairs of points on this curve having the same value of γ and actually represent the same rotating wave, displaying the double valuedness of μ (equal to P_0) discussed in Remark 2.0.3.

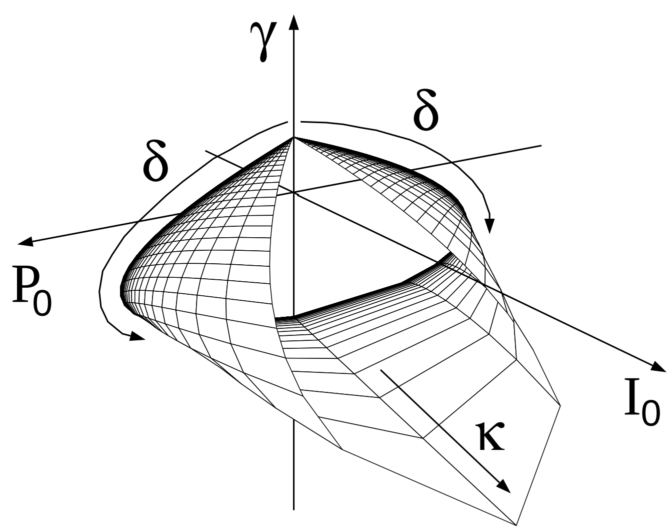

Fig. 3. Part of the surface Σ_T at $m = T = 1$ (for $\kappa < .999$).

The folded structure of Σ_T seen in Figure 3 illustrates the fact that Σ_T cannot be covered by a single coordinate chart in any two of the variables P_0, I_0, and γ, i.e. there are always singularities somewhere if any two are used, leading to the need for the variables μ_1 and μ_2. This folded structure is intimately related to the structure of the potential $V(Q; \mu, \gamma)$ (given by (10)). To see this, consider the level curves of P_0 in the δ, κ simplex some of which are plotted for a particular subfamily of uncentered nonlinear phase waves in Figure 4.

The locations of these level curves are independent of m and T (the overall value of P_0 is simply renormalized). The plot illustrates how every level curve connects two distinct rotating waves, except where the curves collapse to a point, at which point P_0 takes its maximum value in the simplex (the fact that *every* rotating wave along the lower edge of simplex, except where P_0 is maximum, is connected to another with a level curve of P_0 lying entirely in the simplex follows easily from the fact that P_0 is everywhere positive in the simplex, zero on the hypotenuse and at $\delta = 1$, and for each fixed κ has a single local maximum as a function of δ which is monotonically decreasing with κ).

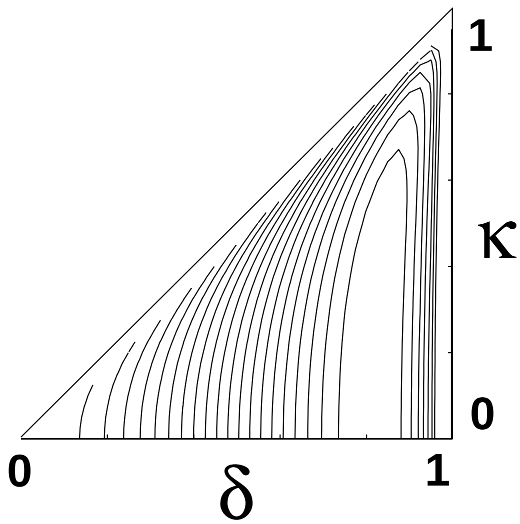

Fig. 4. The level curves of P_0 towards the interior of the simplex.

The deeper reason for the fact that pairs of rotating waves of period T share the same value of P_0 is the specific quadratic-quartic nature of $V(Q;\mu,\gamma)$ for the fcNLS equation. The reader can discover this by explicitly solving the following system of equations for a rotating wave to existence with nearby traveling waves of period T,

$$V'(a;\mu,\gamma) = 0\,, \qquad V''(a;\mu,\gamma) = \frac{8\pi^2 m^2}{T^2}\,. \tag{30}$$

Here, a is the rotating wave amplitude and the function $V(Q = a;\mu,\gamma)$ is given by (10) with parameter $\mu = P_0$. From this discussion it can be seen that the number of rotating waves connected by level curves of P_0, and hence the structure of Σ_T, depends crucially on the nonlinearity chosen for the GNLS equation.

4 Persistence Criteria

This section introduces the Melnikov conditions for persistence, and describes the basic idea behind their implementation.

Various persistence criteria for GNLS traveling waves were given in Propositions 3.0.1, 3.0.2, 4.0.3, 4.0.4, and 4.0.5 in [2] in terms of two "Melnikov" functions. In general, GNLS equations possess traveling wave solutions which are not fully periodic, i.e, their phase profiles $S(z)$ may have periods different from their amplitude profiles $Q(z)$. Because we are only considering fully periodic waves, we work only with the "restricted" Melnikov functions, defined to be the Melnikov functions restricted to the fully periodic traveling waves.

Using (4) and the definitions of μ_1 and μ_2 given by (25) and (26) in this paper and expressions (58) and (59) in [2] with $\mu = P$, the restricted Melnikov conditions on the periodic traveling waves (11–14) or (16–20), are given by

$$W^1(\delta,\kappa,T) = \int_0^T \left(c\mu + \left(\frac{c^2}{2} + \alpha - r\right)Q^2 + 2(1+q)Q^4\right) dz\,, \tag{31}$$

$$W^2(\delta,\kappa,T) = \int_0^T \left(\left(\frac{c^2}{4} + \alpha - r\right)\mu + c\frac{\mu^2}{Q^2} + 2(1+q)\mu Q^2 + c(\partial_z Q)^2\right) dz\,. \tag{32}$$

For uncentered waves, Proposition 3.0.1 of [2] gave necessary and sufficient criteria for a fully periodic uncentered traveling wave to persist as a fully periodic wave of the same period. In this paper, we will use only the neccesary criteria given by Proposition 3.0.1 (condition (60) of that proposition, the vanishing of the Melnikov functions at the candidate traveling wave), and the "weaker" sufficiency criteria of Proposition 3.0.2 of [2], which is stated only in terms of the restricted Melnikov functions. The latter ensures that at least some traveling wave persists with an amplitude profile of the same period (the period of the phase of the persisting wave may still vary). We took this approach because the sufficiency criteria for persistence of a fully periodic wave of the same period, which is given by Proposition 3.0.1 of [2] in terms of the nonrestricted Melnikov functions, is more difficult to analyze but is met generically if the necessary criteria and weaker sufficiency criteria are met.

For uncentered waves, our basic strategy is as follows. Following Proposition 3.0.1 of [2], we set both $W^1(\delta,\kappa,T)$ and $W^2(\delta,\kappa,T)$ equal to zero to obtain neccessary conditions for persistence. Because the number of unknowns is equal to the number of equations, this tends to pick out isolated travelling waves. Each of these equations will generally have zeros along curves in the κ, δ simplex, which we will call "root lines". Our strategy is to locate points where these root lines intersect. This idea is represented schematically in Figure 5. Proposition 3.0.2 of [2] tells us that if the root lines cross transversely, then the NLS traveling wave corresponding to the crossing point persists as a traveling wave in the weaker sense as described above. Therefore the goal is to locate these transverse crossings of the root lines. Note in the figure that the root lines are drawn emanating from a single persisting rotating wave. As we will see in section 6, this is typical and provides a way to find transverse crossings from changes in stability of the rotating wave.

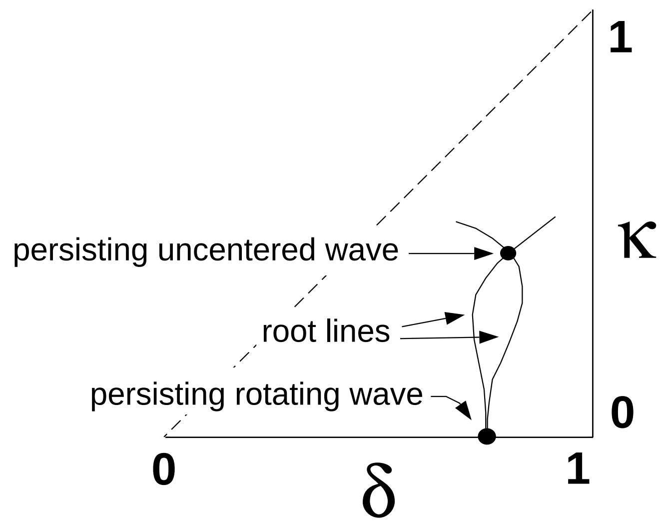

Fig. 5. Schematic diagram illustrating how the "root lines" of the Melnikov conditions pick out persisting uncentered waves

For linear phase waves, we will use both Propositions 4.0.3 and 4.0.5 of [2], which give necessary and sufficient criteria, respectively, for the persistence of *fully* periodic linear phase waves. For the subfamilies of uncentered linear phase waves, the root-line crossing approach described above for uncentered waves can still be used to obtain the weak form of persistence, even though these waves exist along the righthand edge of the simplex ($\delta = 1$). This is because the linear phase subfamilies are embedded in the larger family of uncentered waves. Figure 6 illustrates the idea. For all the linear phase waves, however, only linear phase waves with $n = 0$ can persist, by Propostion 4.0.3 of [2], and on these waves $W^2(\delta, \kappa, T)$ vanishes identically (this is why there is a vertical root line in Figure 6). Thus, if the first Melnikov condition has a zero at one of these waves, then both necessary conditions are satisfied. Moreover, assuming that the Melnikov functions have a zero at a particular wave, only a simple zero of the first Melnikov function is sufficient for persistence of the wave as fully periodic wave of the same period, by Proposition 4.0.5 of [2]. Thus, we need only work with $W^1(\delta, \kappa, T)$ for linear phase waves.

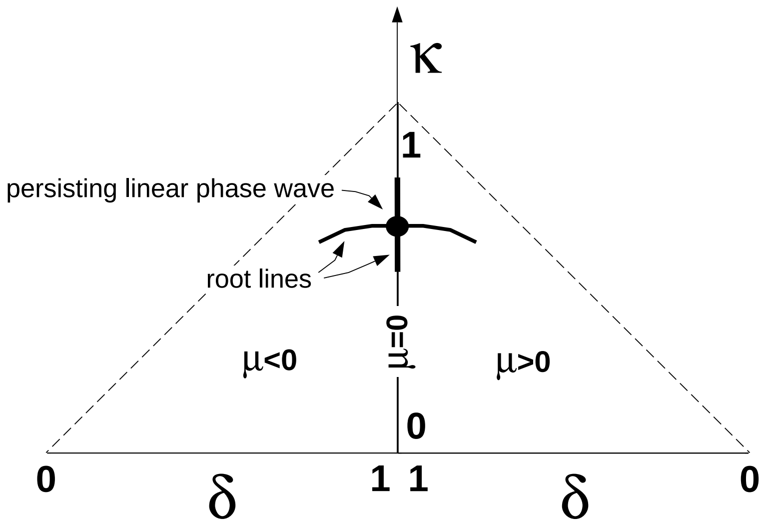

Fig. 6. Schematic diagram illustrating the crossing of root lines for the uncentered linear phase subfamilies.

5 Persistence of fcNLS Linear Phase Waves

In general, it is nontrivial to determine roots of (31–32), i.e. points at which the necessary conditions for persistence are satisfied. In this section we cover an important case which can be analyzed explicitly: the case of the linear phase waves. These consist of the even/odd centered waves (11–14), and the uncentered linear phase waves (24). The case of uncentered nonlinear phase waves is discussed in the next section. In both of these sections, we discuss the case $T = 1$ for simplicity. Results for other periods can be recovered by a simple rescaling (this is only the case when all the nonlinearities in the GCGL equation are of simple power law type with the same power).

The persistence results of this section are closely related to bifurcations in the fcCGL equation with sufficiently small ϵ. We have the following proposition:

Proposition 5.0.1 *(i) Of the fcNLS linear phase waves, only the stationary ($n = c = 0$) waves can persist.*

(ii) In each family (for each m) of period one, stationary, centered waves, a single wave persists as a stationary wave of period one if and only if $r \geq \pi^2 m^2$. Under small perturbation ($0 < \epsilon << 1$), the persisting wave bifurcates from the zero solution in a Hopf bifurcation at $r = \pi^2 m^2 + O(\epsilon)$, which occurs when the zero solution loses stability to sideband perturbations of wavenumber $k_{m/2} = \pi m$.

(iii) In each subfamily (for each m) of period one, stationary, uncentered linear phase waves, a single wave persists as a stationary wave of period one if and only if $r \geq 2m^2 q\pi^2$. Under small perturbation ($0 < \epsilon << 1$), the persisting wave bifurcates from the unique, persisting, period one, rotating wave with $n = 0$ in a Hopf bifurcation at $r = 2m^2 q\pi^2 + O(\epsilon)$, which occurs when this rotating wave loses stability to sideband perturbations of wavenumber $k_m = 2\pi m$.

Proof: The first part of the proposition, the persistence of only stationary waves, follows immediately from Proposition 4.0.3 of [2].

We next note that when the restricted Melnikov function $W^1(\delta, \kappa, T)$, given by (31) is evaluated at $n = 0$ and $\mu = 0$ (and so $c = 0$), then it is exactly the Melnikov function $W^1(A_*)$ specialized to the fcNLS appearing in the statement of Proposition 4.0.3 of [2], so that we may analyze $W^1(\delta, \kappa, T)$ to apply the results of both Proposition 4.0.3 and Proposition 4.0.5 of [2].

Moreover, $W^1(\delta, \kappa, T)$ (for all n, μ and c), can also be written in the form

$$W^1(\delta, \kappa, T) = \int_0^T \Big((\partial_z Q)^2 + Q^2 (\partial_z S)^2 - rQ^2 + 2qQ^4 \Big) \, dz \, . \tag{33}$$

This follows because this latter form can be tranformed into the form (31) by using the relationships

$$\partial_{zz} Q = \frac{\mu^2}{Q^3} - \left(\alpha + \frac{c^2}{4}\right) Q - 2Q^3 \,, \tag{34}$$

$$\partial_z S = \frac{c}{2} + \frac{\mu}{Q^2} \,, \tag{35}$$

and an integration by parts (the first of these relationships follows immediately by differentiating the potential (7) and setting the result equal to $-\partial_{zz} Q$).

Specializing to $n = 0$ and $\mu = 0$ (and so $c = 0$ and hence $\partial_z S = 0$) in (33), the restricted Melnikov condition (31) becomes

$$W^1(\kappa, T = 1) = \int_0^1 -rQ^2 + (\partial_z Q)^2 + 2qQ^4 \, dz \; = 0,$$

where this condition is to be evaluated on the even/odd centered waves (11–14), and the uncentered linear phase waves (24).

By Proposition 4.0.3 of [2], this is a necessary condition for persistence. Supposing this condition is met at some $\kappa = \kappa_*$, then Proposition 4.0.5 of [2] (with the choice $\mu_2 = \kappa$) tells us that a sufficient condition for persistence is that $W^1(\kappa, T = 1)$ has a simple zero with nonvanishing derivative with respect to κ at κ_*. We therefore seek to show that these conditions are met at values of r in accordance with Proposition 5.0.1.

For period one stationary centered waves (SCWs) we have $Q = \lambda\kappa \,\mathrm{cn}(\lambda z, \kappa)$, where $\lambda = 2mK(\kappa)$, or

$$\begin{aligned} 0 &= W^1_{\mathrm{SCW}} \\ &= \int_0^1 -r\lambda^2\kappa^2 \,\mathrm{cn}^2(\lambda z, \kappa) + \lambda^4\kappa^2 \,\mathrm{sn}^2(\lambda z, \kappa) \,\mathrm{dn}^2(\lambda z, \kappa) \\ &\qquad +2q\lambda^4\kappa^4 \,\mathrm{cn}^4(\lambda z, \kappa) \, dz \,. \end{aligned} \tag{36}$$

For period one stationary uncentered linear phase waves (SULPWs) we have $Q = \lambda \,\mathrm{dn}(\lambda z, \kappa)$, where $\lambda = 2mK(\kappa)$, or

$$\begin{aligned} 0 &= W^1_{\mathrm{SULPW}} \\ &= \int_0^1 -r\lambda^2 \,\mathrm{dn}^2(\lambda z, \kappa) + \lambda^4\kappa^4 \,\mathrm{sn}^2(\lambda z, \kappa) \,\mathrm{cn}^2(\lambda z, \kappa) + 2q\lambda^4 \,\mathrm{dn}^4(\lambda z, \kappa) \, dz \,. \end{aligned} \tag{37}$$

By changing variables to $x = \lambda z$ and canceling a factor of $\lambda \kappa^2$ in $W^1_{\rm SCW}$ and a factor of λ in $W^1_{\rm SULPW}$, these become

$$0 = W^1_{\rm SCW} = \int_0^\lambda -r\,\mathrm{cn}^2(x,\kappa) + \lambda^2\,\mathrm{sn}^2(x,\kappa)\,\mathrm{dn}^2(x,\kappa) + 2q\lambda^2\kappa^2\,\mathrm{dn}^4(x,\kappa)\,dx\,, \tag{38}$$

and

$$0 = W^1_{\rm SULPW} = \int_0^\lambda -r\,\mathrm{dn}^2(x,\kappa) + \lambda^2\kappa^4\,\mathrm{sn}^2(x,\kappa)\,\mathrm{cn}^2(x,\kappa) + 2q\lambda^4\,\mathrm{dn}^4(x,\kappa)\,dx\,, \tag{39}$$

respectively.

We now prove that $W^1_{\rm SCW}$ has a single root if and only if $r \geq \pi^2 m^2 = r_c$. It is easily shown by the linear stability analysis that r_c is the value of r at which the zero solution loses stability to sidebands with wavenumber $k_{m/2} = \pi m$ (this is true for all $\epsilon > 0$).

Using the identities

$$\int_0^\lambda \mathrm{sn}^2(x,\kappa)\,\mathrm{dn}^2(x,\kappa)\,dx = (1-2\kappa^2)\int_0^\lambda \mathrm{cn}^2(x,\kappa)\,dx + 2\kappa^2\int_0^\lambda \mathrm{cn}^4(x,\kappa)\,dx\,, \tag{40}$$

$$\int_0^\lambda \mathrm{cn}^4(x,\kappa)\,dx = -\frac{2}{3}\frac{(1-2\kappa^2)}{3\kappa^2}\int_0^\lambda \mathrm{cn}^2(x,\kappa)\,dx - \frac{(\kappa^2-1)}{3\kappa^2}\,, \tag{41}$$

$$\int_0^{K(\kappa)} \mathrm{cn}^2(x,\kappa)\,dx = \frac{E(\kappa)}{\kappa^2} - \frac{(1-\kappa^2)}{\kappa^2}K(\kappa)\,, \tag{42}$$

and the definition of λ we can write

$$W^1_{\rm SCW} = rc_r(\kappa) + c_1(\kappa) + qc_q(\kappa)\,, \tag{43}$$

where

$$c_r(\kappa) = -\frac{2m}{\kappa^2}\Big(E(\kappa) - (1-\kappa^2)K(\kappa)\Big)\,, \tag{44}$$

$$\begin{aligned} c_1(\kappa) = &-\frac{8m^3}{3}\frac{(1-2\kappa^2)}{\kappa^2}K(\kappa)^2\Big(E(\kappa) - (1-\kappa^2)K(\kappa)\Big) \\ &+\frac{16m^3}{3}(1-\kappa^2)K(\kappa)^3\,, \end{aligned} \tag{45}$$

$$\begin{aligned} c_q(\kappa) = &-\frac{32m^3}{3}\frac{(1-2\kappa^2)}{\kappa^2}K(\kappa)^2\Big(E(\kappa) - (1-\kappa^2)K(\kappa)\Big) \\ &+\frac{16m^3}{3}(1-\kappa^2)K(\kappa)^3\,. \end{aligned} \tag{46}$$

By examining the asymptotic behavior of these functions as $\kappa \to 0$ and $\kappa \to 1$, the following facts can be established about these functions, which we state without proof. The function c_r is negative at $\kappa = 0$ and decreases monotonically to $-2m$ at $\kappa = 1$. The function c_1 increases monotonically from the value $(r_c c_r(0))$ at $\kappa = 0$ and diverges as $\kappa \to 1$. The function c_q increase monotonically from 0 at $\kappa = 0$ and diverge as $\kappa \to 1$.

From these facts it follows that $W^1_{\rm SCW}(\kappa = 0)$ is zero at $r = r_c$, and monotonically decreasing in r. It also follows that $W^1_{\rm SCW} \to +\infty$ as $\kappa \to 1$. This establishes that there is at least one root for $r \geq r_c$. We now need to show that this root is simple and unique, and that there are no roots for $r < r_c$.

To show that there are no roots for $r < r_c$, it is sufficient to show that $r_c c_r(\kappa) + c_1(\kappa)$ is monotonically increasing from zero. This can be done, and it immediately follows from the positivity and monotonicity of c_q and c_1 that $W^1_{\rm SCW}$ is positive for all κ in $[0, 1)$ for all $r < r_c$.

To show that the root for $r > r_c$ is unique, it is sufficient to show that $W^1_{\rm SCW}$ can have no local maxima for κ in $(0, 1)$. We prove this by contradiction. Suppose that $W^1_{\rm SCW}$ has an extrema at $0 < \kappa = \kappa^* < 1$. By setting $dW^1_{\rm SCW}(\kappa^*)/d\kappa = 0$, it follows that

$$f(\kappa^*) \equiv \frac{-c_1'(\kappa^*) - q c_q'(\kappa^*)}{c_r'} = r\,,$$

(where primes denote differentiation with respect to κ). If the extrema is a local maximum, then $f(\cdot)$, viewed as a function of κ must be a decreasing function at $\kappa = \kappa^*$. However, it can be checked that the functions $-c_r'/c_r'$ and $-c_q'/c_r'$ are both increasing functions for all κ, so that $f(\cdot)$ must be increasing for any q. It also follows from this that there can only be one local minima.

It can also be shown that $dW^1_{\rm SCW}(0)/dr$ is zero at r_c and monotonically decreasing. Together with the facts above, this implies that $W^1_{\rm SCW}$ is positive and monotonically increasing for all $r < r_c$, and has exactly one local minima in $(0,1)$ for all $r > r_c$.

The fact that the derivative at the unique root for $r > r_c$ follows from the monotonically increasing (as opposed to monotonically nondecreasing) nature of the functions involved.

For the uncentered linear phase waves, we obtain

$$\begin{aligned} W^1_{\mathrm{SULPW}} = & -8m^3\left(\frac{r}{4m^2} + \frac{4q+1}{3}(\kappa^2-2)K(\kappa)^2\right)E(\kappa) \\ & +8m^3\frac{2}{3}(q+1)(\kappa^2-1)K(\kappa)^3 \, . \end{aligned} \tag{47}$$

The proof for this case is very similar to the one above, so we omit it, except to mention that in this case, the selection condition W^1_{SULPW} has a local minima but no roots for $r < r_c = 2qm^2\pi^2$, and is monotonically increasing with a unique root for all $r \geq r_c = 2qm^2\pi^2$. It is again easily shown by the linear stability analysis, for $0 < \epsilon << 1$, that the selected period one rotating wave with $n = 0$ loses stability to sideband perturbations with wavenumber $k_m = 2\pi m$ exactly at $r_c = 2qm^2\pi^2$. Thus, selection begins with the wave bifurcating from the selected rotating wave in a Hopf bifurcation. This concludes the proof of the proposition.

To confirm the proposition numerically, we choose $r = 24$ and $q = 1$, so that $r > 2q\pi^2$ (we consider $m = 1$), and solve numerically for κ_* yielding $\kappa_* = 0.7960$. Therefore, the selected uncentered linear phase wave should be

$$A(x,t) = \lambda\, \mathrm{dn}(\lambda x, \kappa) e^{\lambda^2(2-\kappa^2)t} \, , \tag{48}$$

$$\kappa \approx 0.7960 \, , \qquad \lambda = 2K[\kappa] \approx 3.9916 \, . \tag{49}$$

Figure 7 plots the amplitude of the selected uncentered linear phase wave (49), as well as numerical approximations of a nearby stationary wave of the fcCGL equation for ϵ between 0 and 0.25, at $r = 24$, $q = 1$. The amplitudes of the fcCGL solutions are seen to approach the predicted persisting fcNLS solution as $\epsilon \to 0$. The fcCGL solutions, which are apparently unstable for the smaller values of ϵ, were approximated numerically by finding the corresponding fixed point solutions of an 8 complex mode Galerkin truncation.

6 Persistence of fcNLS Uncentered Nonlinear Phase Waves

In this section we explore the persistence of uncentered fcNLS waves under fcCGL perturbation using the restricted Melnikov functions (31) and (32)

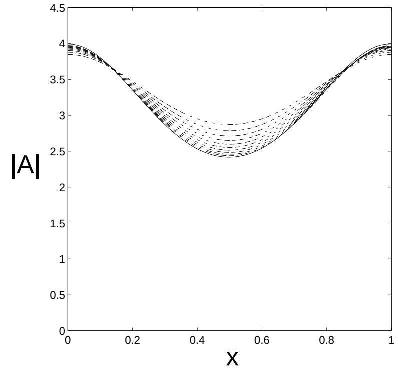

Fig. 7. Numerical approximations of fcCGL stationary wave solutions (dashed lines) for ϵ between 0 and 0.25 at $r = 24$ and $q = 1$ and their limiting fcNLS wave (solid line).

and Proposition 3.0.2 of [2]. As in the preceding section, we examine the persistence of period one fcNLS waves. In the preceding section we proved that for the uncentered linear phase waves, which exist at $\delta = 1$, a single stationary $(n = c = 0)$ wave for each m persists as a period one stationary wave for all $r \geq 2qm^2\pi^2$. We also showed that the beginning of persistence as r is increased above $2qm^2\pi^2$ is associated with a Hopf bifurcation in the weakly perturbed fcNLS equation $(0 < \epsilon << 1)$ wherein the persisting $n = 0$ rotating wave loses stability to sideband perturbations with wavenumber $k_m = 2\pi m$. In this section we prove very similar results for uncentered nonlinear phase waves, and give some numerical results to give a broader picture of traveling wave persistence.

Before we state some propositions, we need to establish some facts about the stability of rotating waves in the fcNLS and fcCGL equations, and the relationship between fcNLS rotating waves and uncentered nonlinear phase waves.

Recall from Remark 2.0.1 that each family of period one uncentered nonlinear phase waves, characterized by a winding number n and modulation number m, reduces to a family of rotating waves (22) of with winding number n as $\kappa \to 0$ (the lower edge of the two δ, κ simplices associated with the family). As described in the remark, the limiting rotating waves (those with $\delta \leq 1$) constitute all the rotating waves with winding number n which are neutrally stable in the unperturbed fcNLS equation to sideband perturbations with wavenumber $k_m = 2\pi m$. For use later, we not that by equation (29) of [2], (with $h(\xi) = -\xi^2$ and $\sigma = -1$) the amplitude a_n of these neutrally stable rotating waves satisfies

$$|a_n|^2 \leq \frac{k_m^2}{4}.$$

We will investigate the persistence of uncentered nonlinear phase waves close to these rotating waves.

Under the fcCGL perturbation, it is easy to show that for each n, only one rotating wave with winding number n will persist with amplitude $a = \sqrt{r/2q}$. Furthermore, for each m and each value of r smaller than a critical value depending on m, n and q, the persisting rotating wave will correspond to a point on the lower edge of each of the two simplices (at the same δ on both simplices). We will henceforth call such a point a 'rotating wave point', and denote it with the coordinate $(\delta_*(r), 0)$ (suppressing the dependence on m, n and q).

We now develope a simple geometric picture of rotating wave stability domains with which our results will be described. In the fcCGL equation, the unique persisting rotating wave with winding number n will have one or more associated two dimensional 'neutral surfaces' in the perturbation parameter space (r, q, ϵ) on which the wave is neutrally stable with respect to wavenumber k_m sideband perturbations. The rotating wave is linearly stable and unstable with respect to wavenumber k_m sideband perturbations, respectively, at points on opposite sides of the neutral surface. For all choices of n and m, the neutral surfaces intersect the plane $\epsilon = 0$ along some neutral curves. We will label such a neutral curve with the symbol $C_{m,n}$.

For a given m and n, the set of neutral curves $C_{m,n}$ partitions the r, q plane into stable and unstable domains, in the sense that if r and q fall in an unstable domain, the persisting rotating wave with winding number n will be linearly unstable with respect to wavenumber k_m sideband perturbations for $0 < \epsilon << 1$ and vice versa.

We have the following proposition:

Proposition 6.0.2 *Let m and n be fixed, and let the two simplices associated with the family of period one uncentered waves with parameters m and n be denoted S^+ and S^-, where S^+ represents the simplex with $(\mathrm{sign}(\mu)n) > 0$ for $\delta < 1$, and S^- represents the simplex with $(\mathrm{sign}(\mu)n) < 0$ for $\delta < 1$. Then if r and q are made to vary such that the point (r, q) transversely crosses any neutral curve $C_(m, n)$ in the r, q plane at some point (r_c, q_c), in a direction such that (a) $m \geq 2n$ and (r, q) passes from a stable domain into an unstable domain, or (b) $m < 2n$ and (r, q) passes from an unstable domain into an stable domain, then an isolated point $(\delta(r, q), \kappa(r, q))$ which satisfies the limited sufficiency criteria of Proposition 3.0.2 of [2] for the persistence of a traveling wave bifurcates from the rotating wave at $(\delta^*, 0)$ at (r_c, q_c) and moves into the interior of S^+. No such point appears for S^-.*

Remark 6.0.4 *By Proposition 3.0.2 of [2], this proposition implies that, un-*

der a sufficiently small fcCGL perturbation, a nonempty set of nontrivial (that is, with $\kappa > 0$) traveling waves with modulation number m and with phase satisfying $S(z = 1) = 2\pi n + O(\epsilon)$ bifurcate from the persisting rotating wave with winding number n in a codimension one bifurcation at (r_c, q_c). These waves are themselves persisting fcNLS waves associated with the simplex S^+. These persisting waves are not necessarily fully periodic because we do not establish here that the phase of the persisting wave maintains a winding number of exactly n over the unit interval. A sufficient condition for this to be so are given in Proposition 3.0.1 of [2]. This condition is met generically, but is not convenient to evaluate for all possible cases.

Remark 6.0.5 *The $n = 0$ case is already covered by Proposition 5.0.1, which asserts that a linear phase uncentered wave persists with unit period beginning with the $n = 0$ rotating wave's loss of stability to wavenumber k_m sidebands. This result was stronger in that we ensured that the persisting wave was fully periodic, and also in that we could analyze persistence away from the initial bifurcation for* all *values of r without any recourse to numerical computation. For the $n \neq 0$ cases it is difficult to obtain as strong a result, but we do give some numerical demonstrations below to show that the restricted sufficiency criteria of Proposition 3.0.2 of [2] is often satisfied for large ranges in r far from the bifurcation.*

Remark 6.0.6 *Takáč [11] has shown that for all $\epsilon > 0$, traveling waves bifurcate from rotating waves of the fcCGL equation when the rotating waves lose stability. In his analysis, the small parameter is the difference $r - r^c$, and therefore the traveling waves are known to exist only for an arbitrarily small range in r. The results of this section establish a similar result in the fcNLS limit and show how this fits into the general picture of the persistence of fcNLS waves. The numerical studies reported here also complement those of Takáč by showing that, at least in the fcNLS limit, persisting traveling waves often exist for large ranges in r, far from the initial bifurcation.*

We now establish Proposition 6.0.2. To do this we look for points in the simplices S^+ and S^- near the rotating waves that satisfy the criteria of Proposition 3.0.2 of [2]. The proposition is established by studying the dependence of these points on r and q. By the discussion in section 4, these are any points $(\delta(r, q), \kappa(r, q))$ in the simplices for which the restricted Melnikov functions $W^{1,2}(\delta, \kappa, T = 1)$ given by (31) and (32) have transversely intersecting 'root-lines'.

As was illustrated in Figure 5, in both S^+ and S^- there are always two root-lines of $W^{1,2}(\delta, \kappa, T = 1)$ that intersect at the (unique) rotating wave point $(\delta_*, 0)$, provided r and q have values such that this point exists. We will prove that the asymptotic directions these two lines emanate from the rotating wave point pass through one another in S^+ as (r, q) crosses a neutral curve $C_{m,n}$

from a stable region to an unstable region, creating a transverse crossing of the root-lines the moves into S^+. To do this, we employ asymptotic expansions of the necessary criteria $W^{1,2}(\delta,\kappa,T=1)=0$ to obtain the asymptotic form of the root-lines near the selected rotating wave point. To determine when the two root-lines cross over one another it turns out to be necessary to expand the Melnikov conditions $W^{(1,2)}(\delta,\kappa)$ to fourth order in κ and δ. Asymptotic expansions for the root-lines, given as the as two curves $\delta_{(1,2)}(k)$ can then be obtained by setting these expansions equal to zero. These straightforward but tedious expansions were carried out with the symbolic software package MATHEMATICA. The root-lines are identical through third order in κ, and to this order have the form

$$\delta_{(1,2)}(k)=\delta_*-\frac{(\delta_*-1)(\delta_*+1)}{4\delta_*}\kappa^2+O_{1,2}(\kappa^4)\,. \tag{50}$$

We see from this that both root-lines bend towards increasing δ in the simplex as they emanate from the rotating wave point for all $\delta_*\in(0,1)$, justifying the schematic picture of Figure 5.

Because the fourth order coefficients differ, it is sufficient to compare these to determine when the root-lines cross over one another as r (δ_*) changes. We find that

$$\left.\left[\frac{d^4\delta_1}{d\kappa^4}-\frac{d^4\delta_2}{d\kappa^4}\right]\right|_{\kappa=0} = \frac{24(1-\delta_*^2)^2m-\mathrm{sign}(\mu)24(2-\delta_*^2)(1-\delta_*^2)^{3/2}n+(1-\delta_*^2)^2\delta_*^2mq}{8\delta_*^5(\delta_*^2-1)^2mq}\,. \tag{51}$$

The denominator of this expression is always positive for $\delta_*\in(0,1)$, so it suffices to consider the sign of the numerator. The root-lines will cross over one another, so that a transverse crossing will bifurcate into or out of the simplex, at values of δ_* for which the numerator has a *simple* zero. Because the first and last terms of the numerator are positive (recall that $m>0$), then the numerator can only change sign as δ_* varies if $\mathrm{sign}(\mu)n>0$. Therefore the bifurcation can only happen in S^+, which establishes the last sentence of Proposition 6.0.2. We henceforth consider only the behavior of the root-lines in S^+.

We now define new parameters

$$R\equiv 2\mathrm{sign}(\mu)n/m\,,\qquad p\equiv\delta_*^2\,. \tag{52}$$

Note that $R>0$ because we consider only cases with $\mathrm{sign}(\mu)n>0$. Due to the signs of the terms in equation (51) for $\delta_*\in(0,1)$ it is easy to show that the numerator has a simple root in $\delta_* \in (0,1)$ *if and only if* the polynomial

$$\Delta(p) = (-4 + 4R^2) + (4 - 4R^2 - 4q)p + (R^2 + 4q - q^2)p^2 + q^2p^3 \qquad (53)$$

also has a simple root in $p \in (0,1)$. We will establish most of our results about bifurcations from this expression. Note that $\Delta(p)$ has at most two extrema, and also that $\lim_{p\to-\infty} \Delta = -\infty$, and $\Delta(1) = R^2 > 0$. The relationship between r, δ_*, and p, which are related monotonically, is given by

$$p = \delta_*^2 = \frac{r - k_n^2}{2qm^2\pi^2} \,. \qquad (54)$$

We now examine two separate cases.

The case $m \geq 2n$ $(0 < R \leq 1)$: If R and q are chosen such that $(4 - 4R^2 - 4q) < 0$ (the coefficient of p in $\Delta(p)$), we have

$$\Delta(0) = -4 + 4R^2 \leq 0 \,, \qquad \left.\frac{d\Delta(p)}{dp}\right|_{p=0} = 4 - 4R^2 - 4q < 0 \,. \qquad (55)$$

These facts, along with those mentioned above, imply that $\Delta(p)$ has exactly one extrema (a local maximum) in $(-\infty, 0)$ and one extrema (a local minimum) in $(0,1)$, and exactly one, simple, root in $(0,1)$. On the other hand, if $(4 - 4R^2 - 4q) > 0$, then along with $R \leq 1$ this implies that $q < 1$, at least, which implies that $(R^2 + 4q - q^2) > 0$ (the coefficient of p^2 in $\Delta(p)$). This implies that $\Delta(p)$ has an extrema in $(-\infty, 0)$, which again implies that $\Delta(p)$ has exactly one extrema in $(0,1)$, and exactly one, simple, root in $(0,1)$.

Let this simple root be called $p = p(q)$, where we have suppressed the dependence of m and n. We now implicitly define a curve $D_{m,n}$ in the r, q plane by substituting $p(q)$ for p in equation (54). This can then be solved for a single valued function $r = r_D(q)$. We see that if r and q are varied such that the point (r,q) crosses $D_{m,n}$ transversely at some (r_c, q_c), then Δ changes sign, and a transverse crossing of the root-lines bifurcates into or out of the simplex S^+. Further examination of $W^{(1,2)}(\delta,\kappa)$, which we omit here, shows that former happens if r is increased from r_c holding $q = q_c$. This latter fact will be used to connect the appearance of the crossing with the change in the rotating wave stability.

The case $m < 2n$ $(R > 1)$: In this case we have

$$\Delta(0) = -4 + 4R^2 > 0 \,, \qquad \left.\frac{d\Delta(p)}{dp}\right|_{p=0} = 4 - 4R^2 - 4q < 0 \,. \qquad (56)$$

The latter fact, along with the fact that $\lim_{p\to-\infty}\Delta(p) = -\infty$, implies that $\Delta(p)$ has at least one extrema in $(-\infty, 0)$. We also have

$$\left.\frac{d\Delta(p)}{dp}\right|_{p=1} = 4 - 2R^2 + 4q + q^2\,, \tag{57}$$

which implies that if q is made sufficiently large the first derivative of $\Delta(p)$ at $p = 1$ will be positive. This derivative being positive, along with the first derivative at $p = 0$ being negative, imply that the one remaining extrema of $\Delta(p)$ will occur in $(0, 1)$, creating the possibility of one (degenerate) or two (simple) roots in $(0, 1)$. That these roots can always be achieved for q sufficiently large follows from the facts that in the numerator on the right-hand side in expression (51), q multiplies a polynomial in δ_* which is positive everywhere for $\delta_* \in (0, 1)$, and that the numerator is always negative at $\delta_* = 0$. On the other hand, if the first derivative of $\Delta(p)$ at $p = 1$ is negative, which can happen for $R < 2$ and q sufficiently small, then $\Delta(p)$ can have no extrema in $(0, 1)$ (because otherwise, there would have to be at least two distinct extrema, which is one too many), and hence $\Delta(p)$ has no roots in $(0, 1)$.

Let the two simple roots be called $p^{(1,2)} = p(q)$, where we have suppressed the dependence of m and n. We now implicitly define a curve $E_{m,n}$ in the r, q plane by substituting $p^{(1,2)} = p(q)$ for p in equation (54) (there is only one curve $E_{m,n}$ because p^1 and p^2 coalesce at some point in the r, q plane, as shown above. In this case, the analysis implies that these can be solved for a single valued function $q = q_E(r)$. We see that if r and q are varied such that the point (r, q) crosses $E_{m,n}$ transversely at some (r_c, q_c), then Δ changes sign, and a transverse crossing of the root-lines bifurcates into or out of the simplex S^+. Further examination of the $W^{(1,2)}(\delta, \kappa)$, which we omit here, shows that former happens for p^1 if r is increased from r_c and for p^2 if r is decreased from r_c while holding $q = q_c$. These facts will also be used to connect the appearance of the crossing with the change in the rotating wave stability.

We now relate the bifurcations found above to the linear stability of the rotating waves under a small fcCGL perturbation. Following [12], we determine the linearized stability of the rotating waves by linearizing the fcCGL equation (3) around the ansatz

$$A = A^{(n)}(1 + p^+(t)e^{ik_m x} + p^-(t)e^{-ik_m x})\,, \tag{58}$$

where $A^{(n)}$ is a rotating wave solution (22) with wavenumber $k_n = 2\pi n$, and $p^+(t)$ and $p^-(t)$ are the complex amplitudes of the sideband perturbation having wavenumber $k_m = 2\pi m$. One finds after linearizing that $p^+(t)$ and

$p^-(t)$ are governed by

$$\frac{d}{dt}\begin{pmatrix} p^{+*} \\ p^- \end{pmatrix} = \begin{pmatrix} -C^{+*} & -2(\epsilon q - \sigma i)|a_n|^2 \\ -2(\epsilon q + \sigma i)|a_n|^2 & -C^{-*} \end{pmatrix} \begin{pmatrix} p^{+*} \\ p^- \end{pmatrix}, \tag{59}$$

where

$$C^{\pm} = (\epsilon + i)k_m^2 \pm 2(\epsilon + i)k_m k_n + 2(\epsilon q + \sigma i)|a_n|^2 . \tag{60}$$

The eigenvalues of the matrix in the system (59) determine the linearized stability of the rotating wave with respect to the mth sideband. Since we are interested in the stability of the rotating waves under an arbitrarily weak fcCGL perturbation, we expand the eigenvalues of this system to first order in ϵ. Using the fact that the selected rotating wave has amplitude $|a_n|^2 = \frac{1}{4}k_m^2 p/\sigma$, we find that the eigenvalue with the largest (most positive) real part, to order ϵ, is given by

$$\begin{aligned} \lambda = & \, i(2k_m k_n + \sqrt{k_m^4(1-p)}) \\ & + \epsilon\left(\left[\frac{k_m|k_n|(2-p)}{\sqrt{1-p}}\right] - \left[k_m^2\left(1 + \frac{1}{2}pq\right)\right]\right) + O(\epsilon^2) . \end{aligned} \tag{61}$$

The leading order term determines the stability of the rotating wave in the fcNLS equation, and is purely imaginary for $p \in (0,1)$. Under small fcCGL perturbation, the rotating wave will be linearly unstable for $p \in (0,1)$ to the mth sideband if and only if the order ϵ term is positive, and stable if negative. From the fact that the two expressions in the square brackets are positive for $p \in (0,1)$, it follows that the order ϵ term will have the same sign as the square of the first expression enclosed in the square brackets minus the square of the second term enclosed in square brackets. If this difference of squares is then multiplied by the positive quantity $\frac{4(1-p)}{k_m^4}$, and the identification $R = \frac{2n}{m}$ is made, this difference becomes exactly $\Delta(p)$, the same condition derived for the occurrence of crossing bifurcations! Therefore, the neutral curves $C_{m,n}$ of the rotating waves are identical with the curves $D_{m,n}$ and $E_{m,n}$ found above, so that transverse crossings appear at the same values (r_c, q_c) at which the stability changes. Furthermore, it is easy to show from (61) and the information given in the two cases above how the crossing appear with variation in r that the crossings always appear with the change of stability as stated in Proposition 6.0.2. This completes the proof of Proposition 6.0.2.

We now attempt to paint a broader picture of the persistence of uncentered waves. We have not made an exhaustive study of the behavior of the root-lines in the simplex, which seems to be quite difficult due to the complexity of the algebraic expressions involved and their dependence on elliptic functions. For example, we cannot rule out the existence of root-lines unrelated to those that intersect at the rotating waves, or that additional crossings to those described so far can be created within the simplex.

To convey a coherent picture, we summarize the results obtained so far along with further information about rotating waves by describing the bifurcation histories of the persisting rotating waves and the associated root-lines and traveling waves under a small fcCGL perturbation ($0 < \epsilon << 1$). The numerical calculations of the root-lines presented presented here carried out with the programming package MATLAB in the 'logarithmic' coordinates

$$\beta = \log(\delta)\,, \qquad \eta = \log(\delta - \kappa)\,,$$

for $\delta < 1/2$, or

$$\beta = \log(1 - \delta)\,, \qquad \eta = \log(\delta - \kappa)\,,$$

for $\delta \geq 1/2$. These coordinates allow accurate tracking of the root-lines even when the lines pass extremely close to the hypotenuse of the κ, δ simplices, where the wave speed diverges.

Case $m \geq 2n$: We assume that m, n, ϵ and q are fixed. As r is increased above $r = k_n^2$, a persisting rotating with winding number n bifurcates from the trivial solution. The rotating wave is initially stable under small fcCGL perturbation with respect to wavenumber k_m sidebands. Numerics and asymptotics in the simplex corners suggest that the root-line associated with W^1 which terminates at the selected rotating wave asymptotically approaches the lower left hand corner of the simplex, while the corresponding root-line associated with W^2 asymptotically approaches the upper right hand corner. The approach to these corners occurs very close to the hypotenuse of the simplex. At a unique value $r = r^c$, where $k_n^2 < r^c < k_n^2 + \frac{1}{2}qk_m^2$, the persisting rotating wave loses stability to the wavenumber k_m sideband perturbations, and nontrivial traveling wave(s) bifurcate from the persisting rotating wave. As mentioned in the remarks following Proposition 6.0.2, one (and only one) of these waves is generically fully periodic with period one. Numerics and asymptotics suggest that the persisting wave appears to remain for all larger values of r (changing shape as r increases). Figure 8 shows the root-lines for this sequence of events for $m = 2$, $n = 1$, and $q = 1$.

Case $m < 2n$: We again assume that m, n, ϵ and q are fixed. As r is increased above $r = k_n^2$, a persisting rotating wave with winding number n bifurcates from the trivial solution. The rotating wave is initially unstable with respect to wavenumber k_m sidebands. Numerics suggest that, typically, the root-lines

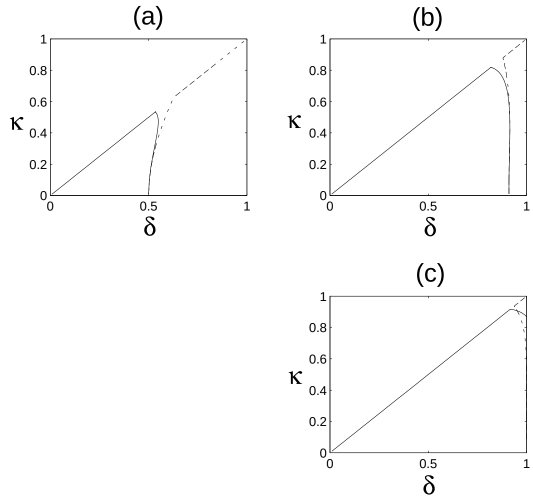

Fig. 8. This figure shows a typical sequence for $m \geq 2n$ wherein a transverse selection crossing is created in the simplex when the selected rotating wave becomes unstable. The W^1 root-line is the solid curve, while the W^2 root-line is dotted. In this example, $m = 2$, $n = 1$, and $q = 1$. In (a), r=59.22, and no internal crossing exists. In (b), r=104.9, the crossing is just being born. In (c), r=150, and the persisting crossing can be seen near the upper right hand corner.

associated with both W^1 and W^2 which terminate at the selected rotating wave asymptotically approach the lower left hand corner of the simplex. As above, the approach to these corners occurs very close to the hypotenuse of the simplex. If q is sufficiently large, or R $(= 2\text{sign}(\mu)n/m)$ is sufficiently close to but greater than 1, the rotating wave will become neutrally stable at some $r = r^{c,1}$, where $k_n^2 < r^{c,1} < k_n^2 + \frac{1}{2}qk_m^2$, and then stable to the mth sideband for a single finite interval in r up to some $r = r^{c,2}$, where $k_n^2 < r^{c,1} \leq r^{c,2} < k_n^2 + \frac{1}{2}qk_m^2$ (this interval will be of zero length if q is chosen such that $\Delta(p)$ has a double root at $r = r^{c,1}$, implying that $r^{c,1} = r^{c,2}$). At $r^{c,1}$ nontrivial traveling waves bifurcate from the rotating wave (with one wave generically of period one). At $r^{c,2}$ nontrivial traveling waves bifurcate out of existence once again. Numerics suggest that these are always the same waves created at $r^{c,1}$. Figure 9 shows this sequence of events for $m = 1$, $n = 1$, and $q = 4.5$. Here, q has been chosen large enough so that there is an interval in r, which is approximately $83.7 < r < 113.9$, for which the selected rotating wave is stable and a crossing in the simplex exists.

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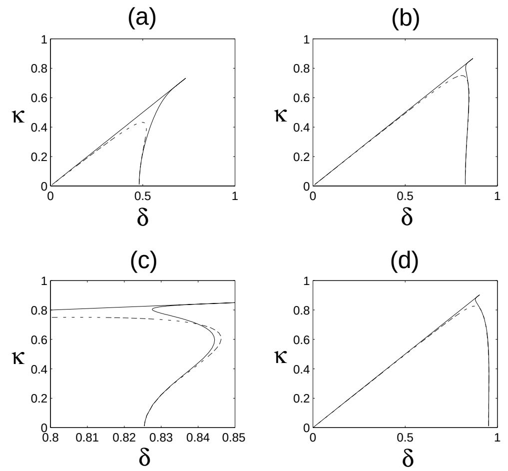

Fig. 9. This figure shows a typical sequence for $m < 2n$ wherein a transverse selection crossing is created in the simplex when the selected rotating wave becomes stable, and then destroyed at a higher value of r when the rotating wave loses stability. The W^1 root-line is the solid curve, while the W^2 root-line is dotted. In this example, $m = 1$, $n = 1$, and $q = 4.5$. In (a), r=60, and no internal crossing exists. In (b), r=100, the crossing exists. This is more clearly visible in (c), which shows a blowup of (b). In (c), r=120, and the crossing no longer exists.

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