Cruz-Pacheco, Gustavo
T-7/CNLS, MS B258

LALP-95-012

CNLS NEWSLETTER

Center for Nonlinear Studies
Los Alamos National Lab.
Los Alamos, NM 87545

No. 114, June 1995

CNLS SCIENCE ACTIVITY

Melnikov Methods for PDEs: Applications to Perturbed Nonlinear Schrödinger Equations

Gustavo Cruz-Pacheco
Program in Applied Mathematics
and
C. David Levermore
Department of Mathematics
Univ. of Arizona

and
Benjamin P. Luce
Center for Nonlinear Studies and Theoretical Division
Los Alamos National Laboratory

ABSTRACT

This article reports our results to date on the development of a kind of Melnikov perturbation technique for PDEs. This technique may be used to determine the persistence of solutions of PDEs possessing at least several conserved quantities when the PDEs are perturbed with additional terms, such as damping and driving terms. Although the ideas behind this technique are of quite general applicability, we specifically focus here on generalized complex Ginzburg-Landau (GCGL) equations as perturbations of generalized nonlinear Schrödinger (GNLS) equations. We report results on almost-periodic, traveling wave, and homoclinic solutions. Complete proofs will appear elsewhere.

1 Introduction

Conservative dynamical systems, including integrable PDEs, are generally structurally unstable to perturbations, for example, those related to damping and driving. This means that much of the phase space structure of the unperturbed system will generally be topologically altered by arbitrarily weak perturbations. For example, elliptic equilibria surrounded by foliations of periodic orbits may be stablized or de-stablized by an arbitrarily weak perturbation, and the nearby periodic orbits destroyed. Moreover, entirely new types solutions may be introduced by the perturbation.

Despite these facts, integrable PDEs such as the nonlinear Schrödinger, Korteweg-deVries, and Sine-Gordon equations, to name a few, are widely used to model natural phenomena such as pulse propagation, water waves, chemical fronts, etc. It is therefore obviously important to inquire as to how solutions of the unperturbed model *persist* when a relevant perturbation is added.

There are several broad classes of techniques/results which have been developed to answer the persistence question, for various types of systems and with various levels of rigor. One class of results, which includes the famous KAM, Moser-Twist, and Aubry-Mather theorems, describes the breakup of tori in integrable systems under perturbation. Another class of methods, which are applicable to integrable PDEs supporting soliton solutions, are various "solitonic perturbation techniques". These techniques usually seek to determine ODEs that describe the evolution of soliton parameters under the influence of perturbations. A third class of techniques, closely related to the method presented in this article, is the Melnikov method for determining the persistence of periodic orbits and homoclinic orbits in integrable ODEs under perturbation. This method is often used to prove the existence of transverse homoclinic orbits for integrable systems under perturbation, which in many cases implies the existence of Smale horseshoes and therefore chaos (for example, see the books by Wiggins [1,2]).

In this article we report on our attempts to answer the persistence question for PDEs which possess at least several conserved quantities (in the absence of perturbation). We present a technique for determining necessary conditions for the persistence of solutions, which is really the infinite-dimensional analogue of the Melnikov method, and demonstrate the technique by using it to find persisting solutions in a particular class of nonlinear PDEs, the generalized complex Ginzburg-Landau (GCGL) equation's

$$\partial_t A = (i + \epsilon)\partial_{xx} A - ih'(|A|^2)A - \epsilon g'(|A|^2)A \,. \tag{1}$$

We consider solutions periodic in x on $[0, 1]$. Here, h and g are real analytic functions on $[0, \infty)$ with at most polynomial growth.

Remark 1.0.1 *Under very general conditions, the GCGL equation is globally well-posed in H^1, has smooth solutions (for initial conditions in H^1), and a compact, and even finite-dimensional attractor (if the PDE possesses so-called inertial manifolds). For example, for power law nonlinearity of the form*

$$h'(\xi) = (\xi)^\sigma, g'(\xi) = -r + q(\xi)^\sigma, \tag{2}$$

where σ is a positive integer, and r and q are positive constants, which includes the cubic NLS and CGL equations (equations (10) and (11) below), these conditions are shown to hold explicitly in [3]. In general, if the functions $h'(\xi)$ and $g'(\xi)$ meet certain conditions, then the above mentioned properties can be explicitly proven to hold by the same methods used in [3]. We shall henceforth assume that these properties hold.

Equation (1) may be written in the form

$$\partial_t A = -i\frac{\delta \mathcal{H}}{\delta A^*} - \epsilon\frac{\delta \mathcal{G}}{\delta A^*}, \tag{3}$$

where the functionals $\mathcal{H}$ and $\mathcal{G}$ are given by

$$\mathcal{H} = \int_0^1 \left(|\partial_x A|^2 + h(|A|^2)\right) dx, \tag{4}$$

and

$$\mathcal{G} = \int_0^1 \left(|\partial_x A|^2 + g(|A|^2)\right) dx. \tag{5}$$

More abstract forms for G could be considered, yielding classes of equations beyond what is sensible to call "Ginzburg-Landau type". We have chosen the Ginzburg-Landau form above because it is general enough to show the utility of the methods contained herein while special enough to keep the number of technical details manageable.

In the limit $\epsilon \to 0$, the GCGL equation (1) reduces to the generalized nonlinear Schrödinger (GNLS) equation

$$\partial_t A = i\partial_{xx} A - ih'(|A|^2)A = i\frac{\delta \mathcal{H}}{\delta A^*}, \tag{6}$$

in Hamiltonian form. The GCGL may therefore be viewed as the damped and (autonomously) driven GNLS equation.

Defining the Poisson bracket of any two functionals $\mathcal{F}$ and $\mathcal{G}$ to be

$$\{\mathcal{F},\mathcal{G}\} \equiv -i \int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A}\frac{\delta\mathcal{G}}{\delta A^*} - \frac{\delta\mathcal{F}}{\delta A^*}\frac{\delta\mathcal{G}}{\delta A} \right) dx\,, \tag{7}$$

the evolution of any functional $\mathcal{F}$ under the GNLS flow is governed by

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F},\mathcal{H}\}\,. \tag{8}$$

Formally, the Hamiltonian $\mathcal{H}$ is conserved by the GNLS flow ($\{\mathcal{H},\mathcal{H}\} = 0$). Besides $\mathcal{H}$, the GNLS flow also conserves the mass $\mathcal{M}$ and the momentum $\mathcal{J}$,

$$\mathcal{M} \equiv \int_0^1 |A|^2\, dx\,, \qquad \mathcal{J} \equiv \frac{1}{2i}\int_0^1 (A^*\partial_x A - A\partial_x A^*)\, dx\,, \tag{9}$$

respectively. These conserved quantities will play an important role in the following.

In the special case of $h(\xi) = \pm\xi^2$, the GNLS equation reduces to the celebrated cubic NLS equation (cNLS),

$$\partial_t A = i\partial_{xx}A \mp 2i|A|^2 A\,, \tag{10}$$

which is completely integrable on the real line and also under periodic boundary conditions ([4,5]). In this case there is an infinite family of conserved functionals in involution with respect to the Poisson bracket (7). The cNLS is widely applied to physical systems. For example, it is the leading order description of pulse propagation in optical fibers [6], which is currently an area of intense research.

The GCGL equation (1) includes the famous cubic Ginzburg-Landau (cCGL) equation

$$\partial_t A = \epsilon r A + (\epsilon + i)\partial_{xx}A - 2(\epsilon q \pm i)|A|^2 A\,. \tag{11}$$

which is obtained by choosing $g(\xi) = -r\xi + q\xi^2$ and $h(\xi) = \pm\xi^2$, where r and q are positive constants. Note that the cCGL reduces to the integrable cNLS as $\epsilon \to 0$. We present some results for this case in this article as a concrete example of our methods. The cCGL equation is important because it is the generic amplitude equation governing the evolution of wavepackets in physical systems near criticality [7]. There are many analytical results for this equation in the literature dealing with the existence and regularity of solutions and other estimates [8-12,3]. Also, the cCGL equation displays low-dimensional long-time behavior [13-17] as evidenced by the existence of an inertial manifold--an exponentially attracting, finite-dimensional manifold that is invariant for positive times.

The question we pose is as follows. For a given choice of $h(\xi)$ and $g(\xi)$, and assuming that some solutions of the GNLS equation (6) for the given choice of $h(\xi)$ are known, which of these solutions *persists* when the perturbation is turned on ($0 < \epsilon << 1$)? The notion of persistence will be precisely defined in section 2. This question can probably never be answered completely for all GNLS equations, because the GNLS may contain an enormous number of different kinds of solutions, and also because of the fact that in many cases we may only be able to obtain necessary conditions for persistence. So far, we have partially answered this question for periodic traveling waves, various trivial cases, and for some types of homoclinic solutions. The first type of solutions are important to understanding wave type behavior, while the third type are central to the study of chaos and bifurcations in PDEs.

The rest of the paper is organized as follows. In section 2 we derive necessary conditions for the persistence of "almost periodic" (in time) solutions of the GNLS equation under the GCGL perturbation, and we apply these criteria to some simple cases. The next two sections discuss traveling wave solutions-the reader interested only in homoclinic solutions may skip these sections and proceed directly to section 5. In section 3, the averaging criteria are applied to traveling wave solutions, and a general theorem is stated. In section 4 these results are explored for a specific case, the cCGL case, and the persistence of traveling waves is linked with changes in stability of the rotating waves. In section 5 the necessary conditions for persistence of GNLS homoclinics connecting rotating waves are given, and as an example, we prove in subsection 5.2 that all homoclinics connecting the so-called rotating wave solutions of GNLS equations are destroyed by GCGL perturbations when the equations have nonlinearities of a certain power law type. We also demonstrate numerically in subsection 5.2 that the GCGL perturbation can introduce new homoclinic orbits.

2 Necessary Criteria for the Persistence of Almost Periodic Solutions

In this section we study the following question, which was motivated in the introduction. For which solutions $A_0(x,t)$ of the GNLS equation (6) are there solutions $A_\epsilon(x,t)$ of the GCGL equation (1) such that when ϵ approaches zero, $A_\epsilon(x,t)$ approaches $A(x,t)$ smoothly? When this happens, we say that the solution $A_0(x,t)$ of the GNLS equation *persists* under the GCGL perturbation.

In this section we use an averaging method to deduce necessary conditions for solutions of the GNLS which are almost periodic (in time) to persist under the GCGL perturbation. We start by stating precisely when a solution of the GNLS equation persists under the GCGL perturbation.

Definition 2.0.1 *We will say that an almost periodic solution $A_0(x,t)$ of the GNLS equation (6) persists under the GCGL perturbation, if and only if there is a family of solutions $A_\epsilon(x,t)$ of the GCGL equation (1) such that when ϵ approaches zero, $A_\epsilon(\cdot,t)$ approaches $A(\cdot,t)$ in the C^∞ topology, uniformally in t.*

We will show that the time average of a particular class of functionals must vanish on an almost periodic GNLS orbit if that orbit persists under the GCGL perturbation. This criterion is derived as follows. Consider a functional F conserved by the GNLS flow. These are real valued functionals of the form

$$\mathcal{F}[A] = \int_0^1 f(A, A^*, \partial_x A, \partial_x A^*, ..., \partial_x^k A, \partial_x^k A^*)\, dx\,,$$

where f is a smooth function of its arguments. In the special case of the cNLS equation (10), f is a polynomial. The time evolution of this functional under the GCGL flow is given by

$$\begin{aligned}\frac{d\mathcal{F}}{dt} &= \int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A^*}\partial_t A + \frac{\delta\mathcal{F}}{\delta A}\partial_t A^*\right) dx \\ &= \int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A^*}\Big(i\frac{\delta\mathcal{H}}{\delta A} - \epsilon\frac{\delta\mathcal{G}}{\delta A}\Big) + \frac{\delta\mathcal{F}}{\delta A}\Big(-i\frac{\delta\mathcal{H}}{\delta A^*} - \epsilon\frac{\delta\mathcal{G}}{\delta A^*}\Big)\right) dx \\ &= -\epsilon\int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A^*}\frac{\delta\mathcal{G}}{\delta A} + \frac{\delta\mathcal{F}}{\delta A}\frac{\delta\mathcal{G}}{\delta A^*}\right) dx\,, \qquad (12)\end{aligned}$$

which does not involve $\mathcal{H}$ because $\mathcal{F}$ is conserved under the GNLS flow. Taking the time average of (12) over the interval $[0,T]$ gives

$$\frac{\mathcal{F}(A_\epsilon(T)) - \mathcal{F}(A_\epsilon(0))}{T} = -\frac{\epsilon}{T}\int_0^T\int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A^*}\frac{\delta\mathcal{G}}{\delta A} + \frac{\delta\mathcal{F}}{\delta A}\frac{\delta\mathcal{G}}{\delta A^*}\right)(A_\epsilon)\,dx\,dt\,.$$

Due to the assumptions on the regularity of solutions and the existence of attractors in section 1, the functional $\mathcal{F}(A_\epsilon(T))$ is necessarily bounded as $T \to \infty$. We therefore have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A^*}\frac{\delta\mathcal{G}}{\delta A} + \frac{\delta\mathcal{F}}{\delta A}\frac{\delta\mathcal{G}}{\delta A^*}\right)(A_\epsilon)\,dx\,dt = 0\,. \qquad (13)$$

We now *assume* the existence of a family of almost periodic solutions of the GCGL equation $A_\epsilon = A_\epsilon(x,t)$ that continuously deform in the C^∞ topology in x to a solution A_0 of the GNLS equation. In other words, for all k

$$\lim_{\epsilon\to 0}\sup_x |\partial_x^k A_\epsilon - \partial_x^k A_0| = 0\,, \qquad \text{uniformally in } t\,. \qquad (14)$$

If we call the integrand in (13) $S_\epsilon(x,t)$, then because of (14),

$$\lim_{\epsilon\to 0} S_\epsilon(x,t) = S_0(x,t) \ \text{ uniformly in } x \text{ and } t\,,$$

which implies

$$\lim_{\epsilon\to 0}\frac{1}{T}\int_0^T\int_0^1 S_\epsilon(x,t)\,dx\,dt = \frac{1}{T}\int_0^T\int_0^1 S_0(x,t)\,dx\,dt, \ \text{ uniformly in } T > 0\,.$$

Hence

$$\begin{aligned} J_{\mathcal{F}}(A_0) &\equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A^*}\frac{\delta\mathcal{G}}{\delta A} + \frac{\delta\mathcal{F}}{\delta A}\frac{\delta\mathcal{G}}{\delta A^*}\right)(A_0)\,dx\,dt \\ &= \lim_{T\to\infty}\lim_{\epsilon\to 0}\frac{1}{T}\int_0^T\int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A^*}\frac{\delta\mathcal{G}}{\delta A} + \frac{\delta\mathcal{F}}{\delta A}\frac{\delta\mathcal{G}}{\delta A^*}\right)(A_\epsilon)\,dx\,dt \\ &= \lim_{\epsilon\to 0}\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A^*}\frac{\delta\mathcal{G}}{\delta A} + \frac{\delta\mathcal{F}}{\delta A}\frac{\delta\mathcal{G}}{\delta A^*}\right)(A_\epsilon)\,dx\,dt \\ &= 0\,. \end{aligned}$$

We summarize all the above into the following proposition.

Proposition 2.0.1 *A necessary condition for the persistence of an almost periodic GNLS solution A_0 in the sense of definition 2.0.1 is the following. For every conserved functional F of the GNLS flow, the time average of $d\mathcal{F}/dt$ at A_0 evolving under the perturbed flow must be zero, i.e.*

$$J_{\mathcal{F}}(A_0) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^1 \left(\frac{\delta \mathcal{F}}{\delta A^*} \frac{\delta \mathcal{G}}{\delta A} + \frac{\delta \mathcal{F}}{\delta A} \frac{\delta \mathcal{G}}{\delta A^*} \right)(A_0)\, dx\, dt = 0\,. \tag{15}$$

It was stated in the abstract and the introduction that the technique which we are presenting, i.e., the necessary conditions just presented, is a type of Melnikov method for PDEs (for an introduction to the Melnikov method for ODEs, see for example the books by Wiggins [1,2]). The connection with the Melnikov method is as follows. The Melnikov method deals with the persistence of periodic orbits and homoclinic orbits of finite-dimensional *integrable* systems. In such systems, the level surfaces of conserved quantities of the unperturbed integrable system, along with a time coordinate, fully parameterize the phase space. This is because each trajectory of the integrable system lies on a unique intersection of level surfaces, one for each conserved quantity. One calculates, to leading order in the strength of the perturbation, the deviation of a perturbed trajectory from these level surfaces. The condition obtained in this way for a periodic orbit or a homoclinic orbit to persist under a perturbation, which is generally a sufficient condition, is that this leading order deviation must vanish separately for every level surface. In other words, to leading order the perturbed trajectory must return to every level surface from whence it started in order to persist. The beauty of the method is that only knowledge of the unperturbed trajectory is required.

The condition given by Proposition 2.0.1 is the exact analogue for infinite-dimensional systems: It is easy to see from the derivation above that condition (15) gives the leading order deviation of the functional $\mathcal{F}$ over the course of the orbit. Unlike the finite-dimensional Melnikov method, it is in general difficult to determine when the necessary condition given above is actually sufficient for the persistence of solutions.

Nonetheless, we shall describe in this article three cases for which the conditions are sufficient. In one of these cases, the finite-dimensional Melnikov method is actually used to prove sufficiency. We also explore a case (for homoclinic orbits) for which sufficiency remains unexplored, but for which an interesting *negative* result is obtained.

The GNLS equations possess so-called rotating wave solutions which are parametrized by a positive amplitude a and an integer n, and have the form

$$A(x,t) = a\exp(i(2\pi nx - \omega t + \phi)), \tag{16}$$

where $\omega = 4\pi^2n^2 + h'(a^2)$. These (trivial) solutions are our first example where the selection criteria prove to be sufficient. Evaluation of the selection criteria for these solutions for the mass functional $\mathcal{M}$ yields

$$0 = J_{\mathcal{M}}(A_0) = 2\Big(4\pi^2n^2 + g'(a^2)\Big)a^2. \tag{17}$$

This necessary condition is also sufficient, because when $g'(a^2) = -4\pi^2n^2$, the rotating wave is an exact solution of the GCGL.

Another simple example where the selection criteria are sufficient is the so-called Lyapunov case. Consider the situation in which $\mathcal{G}$ is itself a conserved functional of the GNLS equation. In this case, the time average of $\mathcal{G}$ according to (12) is given by

$$\frac{d\mathcal{G}}{dt} = -2\epsilon\int_0^1 \left|\frac{\delta\mathcal{G}}{\delta A^*}\right|^2 dx. \tag{18}$$

Hence, $\mathcal{G}$ is decreasing on all solutions of the GCGL equation. If $\mathcal{G} \geq 0$ for all complex-valued functions $A(x)$, periodic in $x \in [0,1]$, then $\mathcal{G}$ may be used formally like a Lyapunov functional. From proposition 2.0.1, one gets

$$\frac{\delta\mathcal{G}}{\delta A^*} = 0. \tag{19}$$

Therefore, the only solutions that can be selected are the critical points of $\mathcal{G}$. However, under the transformation

$$A(x,t) = e^{-irt}\tilde{A}(x - ct, t),$$

the GCGL can be written (only in the Lyapunov case, and suppressing the tilde) as

$$\partial_t A = -(\epsilon + i)\frac{\delta\mathcal{G}}{\delta A^*}. \tag{20}$$

Comparing this GCGL equation with the selection criteria (19), we see that solutions satisfying the selection criteria are also solutions of the GCGL equation. The selection criteria are therefore also sufficient in this case. Note that, in fact, only traveling wave type solutions persist, because the necessary condition implies that $\partial_t A = 0$ for persisting solutions.

3 Persistence of GNLS Traveling Wave Solutions

In this section we consider the persistence of spatially-temporally periodic traveling wave solutions of the GNLS equation (6). By definition, a traveling wave solution $A(x,t)$ of (6) has the form

$$A(x,t) = e^{-i\alpha t} B(x - ct)\,, \tag{21}$$

where B is a periodic, complex-valued function of $z = x - ct$ of period one.

From the fact that B is periodic, it follows from Proposition 2.0.1 that the selection criteria for GNLS traveling waves to persist is

$$\int_0^1 \left(\frac{\delta \mathcal{F}}{\delta A^*} \frac{\delta \mathcal{G}}{\delta A} + \frac{\delta \mathcal{F}}{\delta A} \frac{\delta \mathcal{G}}{\delta A^*} \right) (A_0)\, dz = 0\,,$$

for all conserved functionals $\mathcal{F}$ of the GNLS equation.

In general the only conserved quantities for the GNLS equation are the mass $\mathcal{M}$, momentum $\mathcal{J}$, and the Hamiltonian $\mathcal{H}$, defined in (9) and (4). On the traveling waves the gradients of these three conserved quantities are linearly dependent,

$$\frac{\delta \mathcal{H}}{\delta A^*} = \alpha_0 \frac{\delta \mathcal{M}}{\delta A^*} + c_0 \frac{\delta \mathcal{J}}{\delta A^*}\,,$$

so that it is enough to impose the conditions only with respect to $\mathcal{M}$ and $\mathcal{J}$. Writing A in the form (21), and writing $B = Qe^{iS}$, the selection conditions take the form

$$\int_0^1 \left(\frac{\delta \mathcal{F}}{\delta Q} \frac{\delta \mathcal{G}}{\delta Q} + \frac{1}{Q^2} \frac{\delta \mathcal{F}}{\delta S} \frac{\delta \mathcal{G}}{\delta S} \right) dz = 0\,,$$

where

$$\mathcal{G} = \int_0^1 \left((\partial_z Q)^2 + Q^2 (\partial_z S)^2 + h(Q^2) \right) dz\,.$$

The first and second conditions using $\mathcal{M}$ and $\mathcal{J}$ yield

$$J_{\mathcal{M}} = \int_0^1 \left((\partial_z Q)^2 - Q^2(\partial_z S)^2 + g'(Q^2)Q^2 \right) dz = 0 \,, \tag{22}$$

$$J_{\mathcal{J}} = \int_0^1 \left(Q^2(\partial_z S)^3 - 2Q(\partial_{zz} Q)\partial_z S + (\partial_z Q)^2 \partial_z S - g'(Q^2)Q^2\partial_z S \right) dz = 0 \,, \tag{23}$$

respectively.

The question now arises as to if and when these necessary conditions are actually sufficient. To study this, we must study the system of ODEs for traveling wave solutions, which we now outline.

Substituting the traveling wave profile (21) into the GCGL equation (1), it follows that B solves the profile equation

$$(i+\epsilon)\partial_{zz}B + i\alpha B + c\partial_z B - ih'(|B|^2)B - \epsilon g'(|B|^2)B = 0 \,. \tag{24}$$

The question of the existence of traveling waves for the GCGL reduces to the question of the existence of periodic orbits with period unity of this equation.

Letting $B = Qe^{iS}$, $P = Q^2(\partial_z S - c/2)$, and $R = \partial_z Q$, the profile system becomes

$$\partial_z P = -\frac{\epsilon}{1+\epsilon^2}\left(cP + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2) \right) Q^2 \right) - \frac{\epsilon^2}{1+\epsilon^2} cQR \tag{25}$$

$$\partial_z Q = \frac{\partial H}{\partial R} \,, \tag{26}$$

$$\partial_z R = -\frac{\partial H}{\partial Q} - \frac{\epsilon}{1+\epsilon^2} cR + \frac{\epsilon^2}{1+\epsilon^2}\left(c\frac{P}{Q} + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2) \right) Q^2 \right) \,, \tag{27}$$

where H and the potential V are given by

$$H = \frac{1}{2}\left(R^2 + V(Q,P;\gamma) \right) \,, \quad V(Q,P;\gamma) = \frac{P^2}{Q^2} + \gamma Q^2 - h(Q^2) \,, \tag{28}$$

and where $\gamma = c^2/4 + \alpha$.

In order to more conveniently analyze this system, we change coordinates from (P,Q,R) to action-angle variables (P,I,Θ), where I and Θ are defined by

$$I = \frac{1}{\pi} \int_{Q^{(1)}}^{Q^{(2)}} \sqrt{2H - V(Q)}\, dQ\,, \tag{29}$$

$$\Theta = \frac{2\pi}{T(I,P,\gamma)} \int_{Q^{(1)}}^{Q} \frac{dQ}{\sqrt{2H - V(Q)}}\,, \tag{30}$$

and where $Q^{(1)}(P,I,\gamma,\epsilon)$ and $Q^{(2)}(P,I,\gamma,\epsilon)$ are two consecutive zeros of $2H - V(Q)$. The variable $T(I,P,\gamma)$ is the period of the orbit given by

$$T(I,P,\gamma) = 2 \int_{Q^{(1)}}^{Q^{(2)}} \frac{dQ}{\sqrt{2H(I) - V(Q)}}\,. \tag{31}$$

In the new variables the perturbed profile system becomes

$$\partial_z P = \epsilon f_1(P,I,\Theta,\alpha,c,\epsilon)\,, \tag{32}$$
$$\partial_z I = \epsilon f_2(P,I,\Theta,\alpha,c,\epsilon)\,, \tag{33}$$
$$\partial_z \Theta = \Omega(P,I,\alpha,c) + \epsilon f_3(P,I,\Theta,\alpha,c,\epsilon)\,, \tag{34}$$

where $Q = Q(P,I,\Theta)$, $R = R(P,I,\Theta)$ are implicitly defined and

$$f_1 = -\frac{1}{1+\epsilon^2}\left(cP + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2)\right)Q^2\right) - \frac{\epsilon}{1+\epsilon^2}cQR\,, \tag{35}$$

$$f_2 = -\frac{1}{1+\epsilon^2}\left(\frac{\partial I}{\partial P}\left(cP + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2)\right)Q^2\right) + \frac{\partial I}{\partial R}cR\right) \tag{36}$$
$$- \frac{\epsilon}{1+\epsilon^2}\left(-\frac{\partial I}{\partial P}cQR + \frac{\partial I}{\partial R}\left(c\frac{P}{Q} + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2)\right)Q\right)\right)\,,$$

$$f_3 = -\frac{1}{1+\epsilon^2}\left(\frac{\partial \Theta}{\partial P}\left(cP + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2)\right)Q^2\right) + \frac{\partial \Theta}{\partial R}cR\right) \tag{37}$$
$$+ \frac{\epsilon}{1+\epsilon^2}\left(-\frac{\partial \Theta}{\partial P}cQR + \frac{\partial \Theta}{\partial R}\left(c\frac{P}{Q} + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2)\right)Q\right)\right)\,.$$

When $\epsilon = 0$, the system reduces to dimension 2 (with P a constant parameter), and the solutions of this system, for the most part, are families of periodic orbits corresponding to traveling wave solutions of the GNLS equation. With the requirement of periodic boundary conditions on $[0,1]$, these families are each parameterized by two integers m and n, and two continuous parameters $I_{m,n}$ and $P_{m,n}$.

We have the following general proposition:

Proposition 3.0.2 *A traveling wave solution of the GNLS equation (6) with parameters $P_{m,n} = P_*$ and $I_{m,n} = I_*$ (for some fixed m and n) persists under the GCGL perturbation if P_* and I_* satisfy the following conditions. i) The parameters P_* and I_* solve the following system of equations*

$$M_1(P,I) \equiv \int_0^1 \left(cP + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2) \right) Q^2 \right) dz = 0\,, \tag{38}$$

$$M_2(P,I) \equiv \int_0^1 \left(c\frac{P^2}{Q^2} + \left(\frac{c^2}{2} + \alpha - h'(Q^2) + g'(Q^2) \right) P + cR^2 \right) dz = 0\,. \tag{39}$$

(ii) The Jacobian of $(M_1(P,I), M_2(P,I))$ is nonsingular at (P_*, I_*):

$$\left. \frac{\partial(M_1, M_2)}{\partial(P,I)} \right|_{(P_*,I_*)} \neq 0\,. \tag{40}$$

The proof of this proposition will be reported elsewhere. It involves an extension of the usual Melnikov method for periodic orbits. The reason for the extension is that the system (34) possesses two actions but only one angle.

We make the following important remark:

Remark 3.0.2 *The two functionals $M_1(P,I)$ and $M_2(P,I)$ are linear combinations of the two functionals $J_{\mathcal{M}}$ and $J_{\mathcal{J}}$ of conditions (22) and (23), respectively, and are therefore exactly equivalent to these necessary conditions. Hence, the proposition tells us that these necessary conditions are actually sufficient when the condition (40) is satisfied.*

Thinking geometrically, the sufficient condition is that a traveling wave persists if the curves defined by the conditions $M_1(P_1, I_1, \alpha, c)$ and $M_2(P_1, I_1, \alpha, c)$ in the (P_1, I_1) plane intersect transversely at (P_*, I_*).

4 Persistence of Traveling Waves in the Cubic CGL Equation

To illustrate the necessary and sufficient selection conditions for traveling waves derived in the last section, we give in this section some explicit results for the case of the cCGL equation (11), which is obtained by choosing $g(\xi) = -r\xi + q\xi^2$ and $h(\xi) = \pm\xi^2$, where r and q are positive constants. We study the focusing case,

$$\partial_t A = \epsilon r A - (\epsilon - i)\partial_{xx} A - 2(\epsilon q - i)|A|^2 A\,, \tag{41}$$

obtained by choosing the case with the minus sign in equation (11). In the limit $\epsilon \to 0$, the cCGL reduces to the integrable focusing cNLS equation

$$\partial_t A = i\partial_{xx} A + 2i|A|^2 A\,. \tag{42}$$

The cNLS equation possesses traveling wave solutions with spatial period one of the form $A = Q(x - ct)e^{iS(x-ct)}$, where

$$Q(z) = \lambda\sqrt{\left(\mathrm{dn}^2(\lambda z, \kappa) - 1 + \delta^2\right)}\,, \tag{43}$$

$$S(z) = \int_0^z \left(\frac{c}{2} + \frac{\mu}{Q^2(x)}\right) dx\,, \tag{44}$$

$$c = 4\pi n \pm 4m\Pi\left(\kappa, -\frac{\kappa^2}{\delta^2}\right)\sqrt{(1-\delta^2)\left(1 - \frac{\kappa^2}{\delta^2}\right)}\,, \tag{45}$$

where $m > 0$ and $n \geq 0$ are integers, κ and δ are real parameters, $\lambda = 2m\,\mathrm{K}(\kappa)$ where $\mathrm{K}(\kappa)$ is the complete elliptic integral of the first kind, $\mathrm{dn}(z, \kappa)$ is the Jacobi dnoidal function with modulus κ, $\Pi(\kappa, b)$ is the complete elliptic integral of the third kind, and $\mu = \sqrt{\lambda^6\delta^2(1-\delta^2)(\delta^2 - \kappa^2)}$.

The function $Q^2(z)$ has a maximum value of $\lambda^2\delta^2$ and a minimum value of $\lambda^2(\delta^2 - \kappa^2)$, which is nonnegative, and oscillates m periods in x over the unit interval. The phase $S(z)$ has winding number n.

Hence, as in the general case, the family of cNLS traveling wave profiles is parameterized by the positive integer m, the integer n, the sign $\pm$, two reals (δ and κ), and arbitrary z-translation and phase shift (which we suppress). The parameters δ and κ are constrained to lie within the so-called "simplex" defined by the inequalities

$$0 < \kappa \leq \delta \leq 1\,, \qquad \kappa < 1\,, \qquad \lambda > 0\,. \tag{46}$$

It can be shown that the transformation from the parameters $(\lambda, \kappa, \delta)$ to those of system (34) is nonsingular everywhere within the simplex. We may therefore apply the results of Proposition 3.0.2 directly. Assuming that the conditions (22) and (23) are satisfied for some (κ^*, δ^*), The sufficient condition for persistence turns out to be

$$J = \left.\frac{\partial(J_{\mathcal{M}}, J_{\mathcal{J}})}{\partial(\kappa, \delta)}\right|_{(\kappa^*, \delta^*)} \neq 0,$$

where $J_{\mathcal{M}}$ and $J_{\mathcal{J}}$ are given by (22) and (23), respectively. Therefore, the sufficient condition for a traveling wave with parameters κ and δ to persist is a transversal intersection of the curves defined implicitly by $J_{\mathcal{M}}(\kappa, \delta)$ and $J_{\mathcal{J}}(\kappa, \delta)$ in the simplex.

Remark 4.0.3 *When the modulus κ in the amplitude (43) goes to zero, the corresponding solution to the focusing cNLS (42) continuously deforms to a neutrally stable rotating wave,*

$$A(x,t) \to m\pi\delta e^{i(2\pi n x - (-2m^2\pi^2\delta^2 + 4\pi^2 n^2)t)}. \tag{47}$$

Such a reduction is a generic feature of traveling wave families of the GNLS, and we will shortly give results connected the stability of these rotating waves to persistence of traveling waves.

Remark 4.0.4 *Another important limiting case of (43) is when $\delta \to 1$ while holding all the other parameters fixed. In this case $\mu = 0$ and therefore the phase S is linear in z*

$$S(z) = \frac{c}{2}z, \qquad c = 4\pi n.$$

This makes the corresponding solution to the focusing cNLS (42) a rotating wave modulated by a dnoidal function traveling with velocity c

$$A(x,t) = \lambda\, \mathrm{dn}(\lambda(x - ct), \kappa) e^{i(2\pi n x - (4\pi^2 n^2 - \lambda^2(2-\kappa^2))t)}.$$

This case can be analyzed completely without any recourse to numerics.

We now derive the explicit conditions for the persistence of cNLS traveling waves under cCGL perturbation. Under this perturbation, the functional $\mathcal{G}$ becomes

$$\mathcal{G} = \int_0^1 \left(|\partial_z A|^2 + q|A|^4 - r|A|^2\right) dz. \tag{48}$$

In this case, the two conditions (22)-(23) for the persistence of traveling waves yield

$$J_{\mathcal{M}} = \int_0^1 \left((\partial_z Q)^2 - Q^2(\partial_z S)^2 - rQ^2 + 2qQ^4 \right) dz = 0 ,$$

and

$$J_{\mathcal{J}} = \int_0^1 \left(Q^2(\partial_z S)^2 - 2Q\partial_{zz}Q + (\partial_z Q)^2 + rQ^2 - 2qQ^4 \right) \partial_z S \, dz = 0 ,$$

respectively. Substituting into these conditions the value of $\partial_z S$ in terms of the amplitude Q from equations (43) and (44), we have

$$J_{\mathcal{M}} = \int_0^1 \left(c\mu + \left(\frac{c^2}{2} + \alpha - r \right) Q^2 + 2(1+q)Q^4 \right) dz = 0 , \tag{49}$$

$$J_{\mathcal{J}} = \int_0^1 \left(\left(\frac{c^2}{4} + \alpha - r \right) \mu + c\frac{\mu^2}{Q^2} + 2(1+q)\mu Q^2 + cQ_z^2 \right) dz = 0 . \tag{50}$$

Summarizing these results, we have:

Proposition 4.0.3 *A necessary condition for the persistence of the traveling wave solutions (43)–(44) under the cCGL perturbation (48) is that the two parameters κ and δ must satisfy equations (49)–(50). If the curves in the simplex defined by these conditions intersect transversely, the conditions are sufficient.*

With straightforward analysis of the selection criteria and linear stability analysis of the rotating wave solutions, we also have:

Proposition 4.0.4 *For all cNLS traveling waves for which $\delta = 1$, the only traveling waves that persist are in the families with $n = 0$. Of these, for each family corresponding to a given value of m, a single traveling wave persists if and only if $r \geq 2m^2q\pi^2$. Moreover, the beginning of selection at $r = 2m^2q\pi^2$ occurs when the $n = 0$ rotating wave is neutrally stable to sideband perturbations with wavenumber $k_m = 2\pi m$* under *weak cCGL perturbation* ($0 < \epsilon << 1$), *i.e., the onset of selection begins with a traveling wave bifurcating from the $n = 0$ rotating wave in a kind of pitchfork bifurcation.*

To confirm Proposition 4.0.4 numerically, we fixed $r = 24$, $m = 1$, and $q = 1$ (so that $r > 2q\pi^2$), and solved numerically the value of κ corresponding to the persisting traveling wave, which yielded $\kappa = 0.7960$. Therefore the selected standing wave should be

$$A(x,t) = \lambda \,\mathrm{dn}(\lambda x, \kappa) e^{\lambda^2(2-\kappa^2)t}, \quad \kappa = 0.7960\,, \quad \lambda = 2K[\kappa] = 3.9916\,. \tag{51}$$

Figure 1 graphs the amplitude of the dnoidal solution (51), as well as numerical simulations of dnoidal like cCGL solutions for ϵ between 0 and 0.25, at $r = 24$, $q = 1$. The amplitudes of the dnoidal like solutions approach the predicted dnoidal cNLS solution very nicely as $\epsilon \to 0$. The cCGL solutions were approximated numerically by finding the corresponding fixed point solutions of an 8 complex mode Galerkin truncation of the cCGL equation, and then reconstructing the solution.

With tedious power series expansions of the curves defined by the selection criteria in the simplex, and by asymptotic analysis of the linear stability of rotating waves in the cNLS limit, we can establish a more general result:

Proposition 4.0.5 *A traveling wave of the focusing cNLS equation with indices n and m must* always *become selected or deselected under a weak cCGL perturbation at values of $r = r^c$ for which the n^{th} rotating wave, under weak cCGL perturbation, undergoes a change in linear stability with respect to sideband perturbations with wavenumber k_m as r is varied (all other parameters fixed). This occurs via a pitchfork bifurcation of the traveling wave with the rotating wave.*

Remark 4.0.5 *The $n = 0$ case is already covered by Proposition 4.0.4. In that particular case, our results are slightly stronger: we could analyze persistence away from the pitchfork bifurcation (for all r) without any recourse to numerical computation. In the general case, it is difficult to do so (except near the bifurcation with the rotating wave), but numerical demonstrations that the selection conditions are satisfied for certain traveling waves for large ranges in r are easily carried out to arbitrary precision, yield positive results, and could probably be extended without much trouble into computer assisted proofs. We have not felt it necessary to carry out such a tedious extension. A numerical example is given below.*

Remark 4.0.6 *Takáč [12] has shown that for all $\epsilon > 0$, a traveling wave with phase winding number n and m spatial oscillations in the unit interval is created whenever the n^{th} rotating wave loses stability with respect to the m^{th} sideband at a value $r = r^c$. In this analysis, the small parameter is the difference $r - r^c$, and therefore this result holds only for a very small range in r. Our results extend and complements Takáč's result by showing that, at least near the cNLS limit, the selected traveling waves persist for large and even in-*

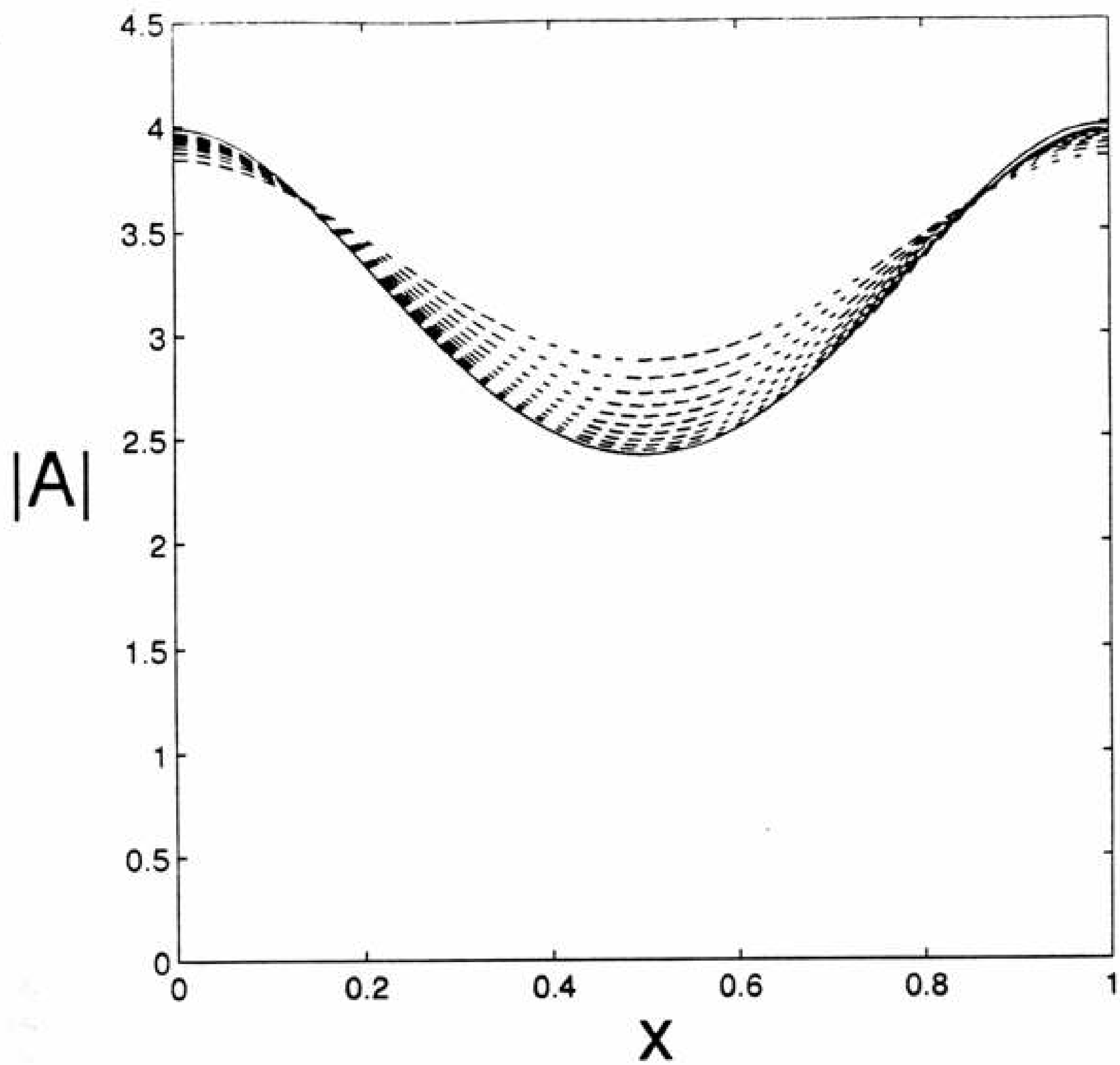

Fig. 1. Numerical simulation of the dnoidal like cCGL solutions (dashed lines) for ϵ between 0 and 0.25 at $r = 24$ and $q = 1$; showing that they approach the dnoidal cNLS solution (solid line) as $\epsilon \to 0$.

finite ranges in r, far removed from the pitchfork bifurcation. The results given by Propositions 4.0.4 and 4.0.5 also establish beyond doubt that the traveling waves found by Takáč are parametrically related to cNLS traveling waves.

To demonstrate Proposition 4.0.5 numerically, we calculate the root lines of the selection conditions numerically. Figure 2 shows the root lines in the simplex, calculated numerically, for $n = 1$, $m = 2$, for three different values of r. At $r = 59.22$, the root lines intersect exactly at the selected rotating wave located at $(\delta, \kappa) = (.5, 0)$, and nowhere else. At $r = 104.9$, the rotating wave is neutrally stable, the root lines are tangent at the rotating wave point. At this point a crossing begins to move into the simplex. At $r = 150$, the crossing

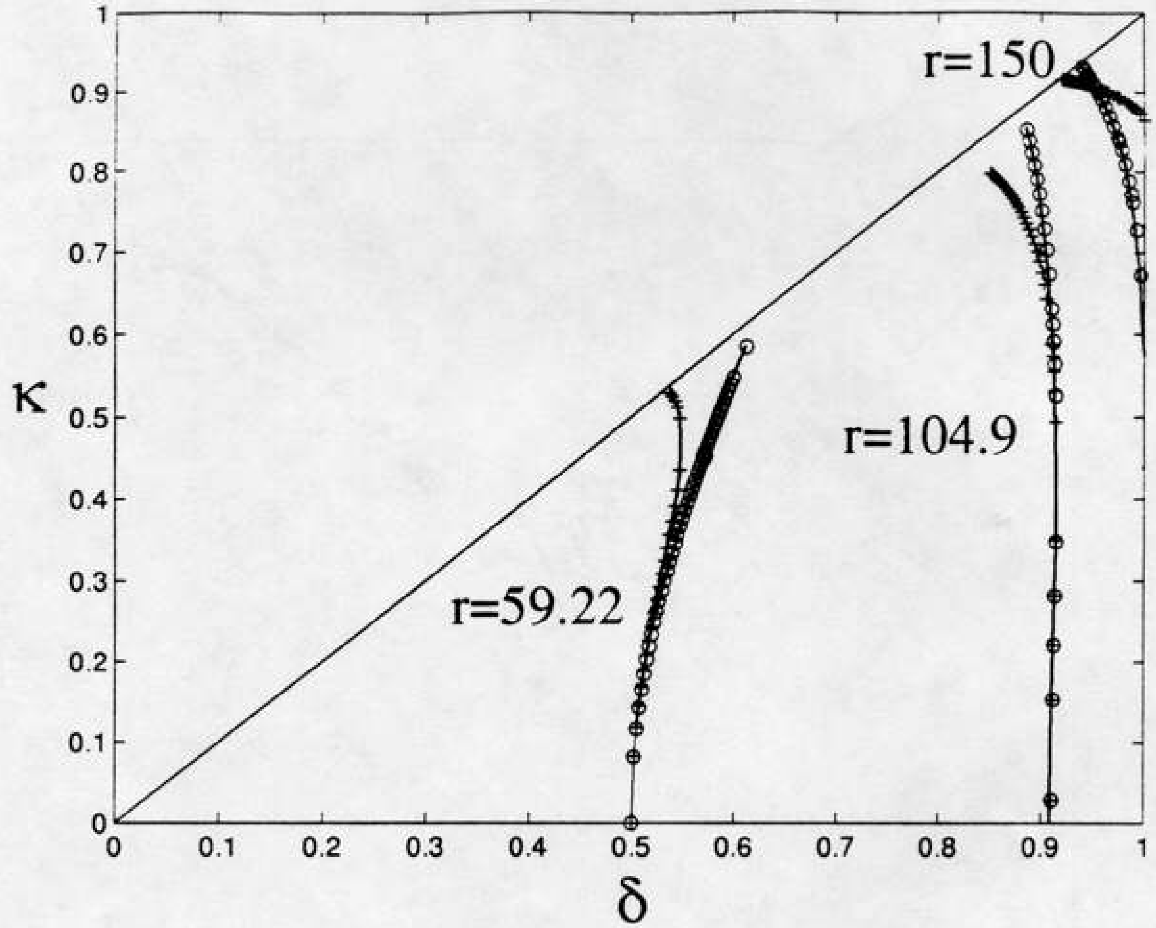

Fig. 2. This figure shows the root lines of the selection criteria for $q = 1$ and different values of r. The circles give numerically calculated values for the full conditions (the solid lines are fit graphically to the lines implied by the circles). For low values of r the curves do not intersect. At $r \approx 104.9$, which is exactly the value at which the selected rotating wave loses stability, a crossing is generated and moves into the simplex as r is increased. For $r = 150$ the (transversal) crossing can still be found in the interior of the simplex.

still persists and can be clearly seen.

5 Necessary Criteria for the Persistence of Homoclinic Solutions

5.1 General Criteria

In this section we present modified but analogous criteria to those derived in section 2 for the persistence of homoclinic solutions of the GNLS equation under a GCGL perturbation.

In the case of the cNLS equation (10), which is integrable, it is known that rotating waves unstable with respect to the m^{th} sideband perturbation form the endpoints of spatially periodic, temporally homoclinic orbits [18,19]. In the general case, for which the GNLS is not integrable, it is less clear as to when there are homoclinic orbits. Some cases may contain many homoclinics, others none at all. In any case, in this section we develop a general criteria, which we later apply to the GNLS and GCGL equations with power law type nonlinearities, which includes as a subcase the important cNLS and cCGL equations.

Let $A_0(x,t)$ be a solution of the GNLS equation which is homoclinic to a rotating wave $A^{(n)}(x,t) = ae^{i(2\pi nx-\omega t+\phi)}$. Clearly, a prerequisite for $A_0(x,t)$ to persist under the GCGL perturbation is for $A^{(n)}(x,t)$ to persist. Assume that $A_\epsilon(x,t)$ is a family of homoclinic orbits for the GCGL equation that continuously deform in the C^∞ topology in x to the GNLS homoclinic orbit $A_0(x,t)$ as $\epsilon \to 0$. The time evolution of any GNLS conserved functional $\mathcal{F}$ evaluated at $A_\epsilon(x,t)$ is given by (equation (12))

$$\frac{d\mathcal{F}}{dt} = \epsilon \int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A^*}\frac{\delta\mathcal{G}}{\delta A} + \frac{\delta\mathcal{F}}{\delta A}\frac{\delta\mathcal{G}}{\delta A^*}\right)(A_\epsilon)\,dx.$$

Because the value of any functional at the beginning and end of a homoclinic orbit are equal, integration of the time evolution of $\mathcal{F}$ gives

$$0 = \frac{\mathcal{F}(+\infty) - \mathcal{F}(-\infty)}{\epsilon} = \int_{-\infty}^{+\infty}\int_0^1 \left(\frac{\delta\mathcal{F}}{\delta A^*}\frac{\delta\mathcal{G}}{\delta A} + \frac{\delta\mathcal{F}}{\delta A}\frac{\delta\mathcal{G}}{\delta A^*}\right)(A_\epsilon)\,dx dt\,. \qquad (52)$$

Call $S_\epsilon(x,t)$ the integrand in the above integral (52). As in the first section, $S_\epsilon(x,t) \to S_0(x,t)$ uniformly in x and t. Define

$$E_\epsilon(t) = \int_0^1 S_\epsilon(x,t)\,dx\,.$$

Assume also $E_\epsilon(t) \to E_0(t)$ in $L^1(-\infty,\infty)$ in t. This implies

$$\lim_{\epsilon\to 0} \int_{-T}^{T} E_\epsilon \, dt = \int_{-T}^{T} E_0 \, dt \quad \text{uniformly in } T\,,$$

because

$$\left| \int_{-T}^{T} E_\epsilon \, dt - \int_{-T}^{T} E_0 \, dt \right| \le \int_{-\infty}^{\infty} |E_0 - E_\epsilon| \, dt \to 0 \quad \text{as } \epsilon \to 0\,.$$

Therefore, it is possible to interchange limits and integration in the following way

$$\begin{aligned} J_{\mathcal{F}}(A_0) &\equiv \lim_{T\to\infty} \int_{-T}^{T} E_0 \, dt \\ &= \lim_{T\to\infty} \lim_{\epsilon\to 0} \int_{-T}^{T} E_\epsilon \, dt \\ &= \lim_{\epsilon\to 0} \lim_{T\to\infty} \int_{-T}^{T} E_\epsilon \, dt \\ &= 0\,. \end{aligned}$$

This gives the following necessary condition for the persistence of a given GNLS homoclinic A_0 under a GCGL perturbation:

$$J_{\mathcal{F}}(A_0) \equiv \int_{-\infty}^{\infty} \int_{0}^{1} \left(\frac{\delta\mathcal{F}}{\delta A^*} \frac{\delta\mathcal{G}}{\delta A} + \frac{\delta\mathcal{F}}{\delta A} \frac{\delta\mathcal{G}}{\delta A^*} \right)(A_0) \, dx dt = 0\,. \tag{53}$$

5.2 Nonpersistence of GNLS Homoclinics

We will use the criteria derived above (53) to prove the following proposition:

Proposition 5.2.1 *All homoclinic solutions connecting rotating wave solutions of the GNLS equation with power law nonlinearity ($h(\xi) = -\frac{2}{s}\xi^s$), i.e.*

$$\partial_t A = i\partial_{xx} A + 2i|A|^{2(s-1)} A\,, \tag{54}$$

are destroyed by any GCGL perturbation generated with the function $g(\xi) = \frac{2q}{p}\xi^p - r\xi$, where p is an integer independent of s and r and q are arbitrary *real numbers.*

For example, with $s = 2$, Proposition 5.2.1 tells us that all the homoclinic solutions of the focusing cNLS equation (42) which are proven to exist in [18,19], are destroyed by the class of GCGL perturbations above.

We explicitly give here the proof of Proposition 5.2.1. To prove that the homoclinic orbits do not persist, it is enough to show that the necessary condition does not hold for a specific functional $\mathcal{F}$. For simplicity we choose the mass functional

$$\mathcal{M} = \int_0^1 |A|^2 \, dx .$$

Hence, the selection criterion (53) becomes

$$J_{\mathcal{M}}(A_0) \equiv \int_{-\infty}^{+\infty} \int_0^1 \left(|\partial_x A_0|^2 + 2q|A_0|^{2p} - r|A_0|^2 \right) \, dx dt = 0 , \tag{55}$$

where A_0 is a candidate GNLS homoclinic orbit. A prerequisite for this integral to vanish is that the rotating wave to which the homoclinic orbit A_0 is associated also satisfies the selection criterion (is a persistent solution). Otherwise, the integrand would be finite as $t \to \pm\infty$, and the integral would diverge. Another way to arrive at this is simply the obvious requirement that the rotating wave associated with the homoclinic must persist if the homoclinic is to persist! In any case, this yields

$$g'(a^2) = 2qa^{2p-2} - r = -4\pi^2 n^2 .$$

Observe that we can write the selection condition (53) in terms of the Hamiltonian $\mathcal{H}$, of the mass $\mathcal{M}$, the L^{2p} norm, and the L^{2s} norm of A_0, in the following way:

$$J(A_0) = \int_{-\infty}^{+\infty} \left(\mathcal{H} - r\mathcal{M} + 2q \int_0^1 |A_0|^{2p} \, dx + \frac{2}{s} \int_0^1 |A_0|^{2s} \, dx \right) dt .$$

The mass and the Hamiltonian are conserved quantities, therefore, they can be evaluated at the end points of the homoclinic, i.e., at the rotating wave. In this way $J(A_0)$ becomes

$$J_{\mathcal{M}}(A_0) = \int_{-\infty}^{+\infty} \left(4\pi^2 n^2 a^2 - \frac{2}{s} a^{2s} - ra^2 + 2q \int_0^1 |A_0|^{2p} \, dx + \frac{2}{s} \int_0^1 |A_0|^{2s} \, dx \right) dt .$$

Now, Hölder's inequality gives

$$\int_0^1 |A_0|^{2s}\,dx \geq \left(\int_0^1 |A_0|^2\,dx\right)^s ,$$

where the equality is hold only if $|A_0|$ is constant in x. Using this last inequality (for both integrals) and the fact that for all homoclinic orbits $|A_0|$ is not constant in x, the following inequality is obtained

$$\begin{aligned}
J_{\mathcal{M}}(A_0) &= \int_{-\infty}^{+\infty} \left(4\pi^2 n^2 a^2 - \frac{2}{s}a^{2s} - ra^2 + 2q\int_0^1 |A_0|^{2p}\,dx + \frac{2}{s}\int_0^1 |A_0|^{2s}\,dx\right) dt \\
&> \int_{-\infty}^{+\infty} \left(4\pi^2 n^2 a^2 - \frac{2}{s}a^{2s} - ra^2 + 2q.\mathcal{M}^p + \frac{2}{s}.\mathcal{M}^s\right) dt \\
&= \int_{-\infty}^{+\infty} \left(4\pi^2 n^2 a^2 - \frac{2}{s}a^{2s} - ra^2 + 2qa^{2p} + \frac{2}{s}a^{2s}\right) dt \\
&= 0 .
\end{aligned} \tag{56}$$

Therefore no homoclinic trajectory of the GNLS equation (54) with simple power law nonlinearities (and linear pumping) terminating at a rotating wave can persist under the GCGL perturbation generated by $g(\xi) = \frac{2q}{p}\xi^p - r\xi$.

This is not the end of the story, however. Something special can happen if the parameters in the perturbation are chosen such that a rotating wave is selected which is *neutrally stable* under weak GCGL perturbation. If homoclinic orbits exist in the unperturbed GNLS equation, such rotating waves always lie at the endpoints of families of homoclinic orbits, where the family shrinks to zero size. Although these "critical" rotating waves do not possess homoclinic orbits, the degeneracy associated with these rotating waves (neutral stability under perturbation), suggests that some delicate and complicated structure, including homoclinic orbits, may be created near the critical rotating waves under the GCGL perturbation.

In fact, we can give a numerical example of a case where a homoclinic orbit is *created* out of the critical rotating wave by the perturbation: For the focusing cNLS equation (42), the $n = 0$ rotating wave has homoclinic orbits related to instabilities in the $m = 1$ direction for $a_0 > \pi$ (there are additional homoclinics as well at higher amplitudes- see [18,19] for derivations, explicit expressions, etc). The perturbation we consider is given explicitly by the cCGL equation (11), which is parameterized by the parameters r, q and ϵ. In the following, we fix $q = 1/2$. Numerically, we have located a homoclinic orbit of the *cCGL equation*, which exists along a one-dimensional curve in the

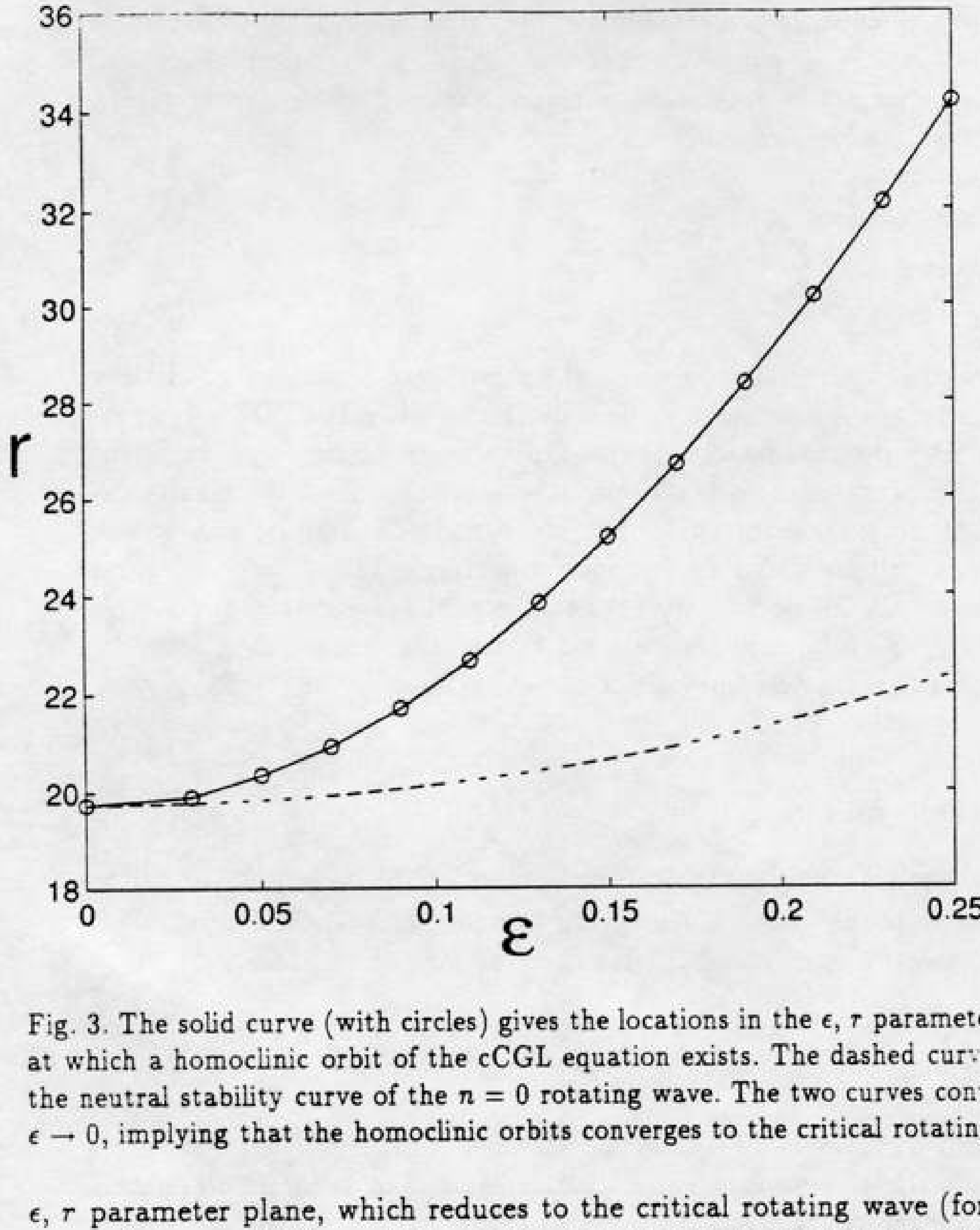

Fig. 3. The solid curve (with circles) gives the locations in the ϵ, r parameter plane at which a homoclinic orbit of the cCGL equation exists. The dashed curve shows the neutral stability curve of the $n = 0$ rotating wave. The two curves converge as $\epsilon \to 0$, implying that the homoclinic orbits converges to the critical rotating wave.

ϵ, r parameter plane, which reduces to the critical rotating wave (for which $a_0 = \pi$) as $\epsilon \to 0$. Figure 3 shows the numerically obtained curve, which was found by "shooting" experiments to locate the homoclinic orbit. In the Figure, therefore, the convergence of the cCGL homoclinic to the critical rotating wave is implied by the convergence of the homoclinic curve (solid line) to the neutral curve of the $n = 0$ rotating wave (dashed line).

Remark 5.2.1 *This family of codimension 1 homoclinic orbits in the cCGL equation is slightly different structurally from the cNLS homoclinics associated* *with the $n = 0$ rotating wave, which were destroyed by the perturbation: whereas the cNLS homoclinics leave and return to the cNLS rotating waves along directions associated with instabilities in the k_1 wavenumber direction, the cCGL homoclinics leave along a k_1 direction but return along the k_0 direction-the direction of pure amplitude perturbations of the rotating wave, which became a stable direction for the rotating wave under the cCGL perturbation. This is in fact what one would expect generically-a homoclinic orbit will generally return along the weakest stable direction.*

6 Conclusion

We have presented a technique by which the persistence of solutions of PDE's possessing conserved quantities may be determined when the PDE's are perturbed. We have shown that this method provides necessary and sufficient criteria for the persistence of traveling wave solutions. Detailed results using this technique were reported for the cNLS and cCGL equations. It was also proven that all homoclinic solutions connecting rotating wave solutions are destroyed by GCGL perturbation when the nonlinearities are of a simple power law type. Finally, a numerical result was reported showing that the perturbation can create new homoclinic structure.

7 Acknowledgements

We would like to thank David McLaughlin, Donald Stark, Karla Horsch, Gregor Kovačič, Tasso Kaper, and Nicholas Ercoloni for useful discussions.

References

[1] S. Wiggins. *Global Bifurcations and Chaos*, volume 73 of *Applied Mathematical Series*. Springer-Verlag, 1988.

[2] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, volume 2 of *Texts in Applied Mathematics*. Springer-Verlag, 1990.

[3] David C. Levermore and Marcel Oliver. The complex Ginzburg-Landau equation as a model problem. *AMS Lectures in Applied Mathematics*. to appear 1995.

[4] V.E. Zakharov and A.B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP*, 34:62–69, 1972. english translation of 1971 article.

[5] M. G. Forest and J. E. Lee. Geometry and modulational theory for the periodic nonlinear schrödinger equation. *AMS Volumes in Applied Mathematics*, 2:35–69.

[6] A. Hasegawa and F. Tappert. *Applied Physics Letters*, 23(142):171, 1973.

[7] A. C. Newell. Envelope equations in nonlinear wave motion. In *Lectures in Applied Mathematics*. AMS, 1972.

[8] C. R. Doering, J. D. Gibbon, and C. D. Levermore. Weak and strong solutions of the complex Ginzburg-Landau equation. *Physica D*, 71:285–318, 1994.

[9] C. R. Doering, J. D. Gibbon, D. D. Holm, and B. Nicolaenko. Low-dimensional behaviour in the complex Ginzburg-Landau equation. *Nonlinearity*, 1:279–309, 1988.

[10] J. M. Ghidaglia and B. Héron. Dimension of the attractors associated to the Ginzburg-Landau partial differential equation. *Physica*, 28D:282–304, 1987.

[11] A. Doelman and E. S. Titi. Regularity of solutions and the convergence of the Galerkin method in the Ginzburg-Landau equation. *Numerical Functional Analysis and Optimization*, 14(3,4):299–321, 1993.

[12] P. Takáč. Invariant 2-tori in the time-dependent Ginzburg-Landau equation. *Nonlinearity*, 5(2):289–321, 1992.

[13] H. T. Moon, P. Huerre, and L. G. Redekopp. Three-frequency motion and chaos in the Ginzburg-Landau equation. *Phys. Rev. Letters*, 49(7), 1982.

[14] L. R. Keefe. Dynamics of perturbed wavetrain solutions to the Ginzburg-Landau equation. *Stud. Appl. Math.*, 73:91–153, 1985.

[15] J. D. Rodriquez and L. Sirovich. Low-dimensional dynamics for the complex Ginzburg-Landau equation. *Physica D*, 43:77–86, 1990.

[16] A. Doelman. Finite-dimensional models of the Ginzburg-Landau equation. *Nonlinearity*, 4(231-250), 1991.

[17] B. P. Luce. Homoclinic explosions in the complex Ginzburg-Landau equation. *Physica D*, 83:1–29, 1995.

[18] N. Ercolani, M. G. Forest, and D. W. McLaughlin. Geometry of the modulational instability, part two, global results. in preparation.

[19] M. J. Ablowitz and B. M. Herbst. On homoclinic structure and numerically induced chaos for the nonlinear Schrodinger equation. *SIAM J. Appl. Math.*, 50(2):339–351, 1990.