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PROPERTIES OF SOME BREATHER SOLUTIONS OF A NONLOCAL DISCRETE NLS EQUATION*

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Abstract. We present results on breather solutions of a discrete nonlinear Schrödinger equation with a cubic Hartree-type nonlinearity that models laser light propagation in waveguide arrays that use a nematic liquid crystal substratum. A recent study of that model by Ben et al [R.I. Ben, L. Cisneros Ake, A.A. Minzoni, and P. Panayotaros, Phys. Lett. A, 379:1705C-1714, 2015] showed that nonlocality leads to some novel properties such as the existence of orbitaly stable breathers with internal modes, and of shelf-like configurations with maxima at the interface. In this work we present rigorous results on these phenomena and consider some more general solutions. First, we study energy minimizing breathers, showing existence as well as symmetry and monotonicity properties. We also prove results on the spectrum of the linearization around one-peak breathers, solutions that are expected to coincide with minimizers in the regime of small linear intersite coupling. A second set of results concerns shelf-type breather solutions that may be thought of as limits of solutions examined in [R.I. Ben, L. Cisneros Ake, A.A. Minzoni, and P. Panayotaros, Phys. Lett. A, 379:1705C-1714, 2015]. We show the existence of solutions with a non-monotonic front-like shape and justify computations of the essential spectrum of the linearization around these solutions in the local and nonlocal cases.

Keywords. discrete nonlinear Schrödinger equations; nonlocal effects; localized solutions; breather solutions; linear stability.

AMS subject classifications. 37K60; 35Q55; 47J10; 47J30; 78A60.

1. Introduction

We present results on breather solutions of a nonlocal discrete nonlinear Schrödinger equation with a cubic Hartree-type nonlinearity introduced by Fratalocchi and Assanto [7] to study the propagation of laser light in waveguide arrays built from a nematic liquid crystal substratum. Further experimental work on this system is reported in [8, 10, 29]. Breather solutions of discrete NLS equations have both theoretical and experimental importance and have been studied extensively, especially for the discrete NLS equation with power nonlinearities, see [16] and references. Recently Ben *et al* [1] reported some interesting new properties of breather solutions of the nonlocal discrete NLS system of [7] in a 1-D lattice, specifically the presence of internal modes around orbitaly stable breathers, and the appearance of front-like solutions that attain their maximum at the interface. The present work studies some of these properties from a rigorous point of view, and includes new related results on breathers of both the local and nonlocal discrete cubic NLS equations.

We will show existence and stability results for two types of solutions of the Fratalocchi-Assanto [7] system. First, we study breather solutions that minimize the Hamiltonian (energy) over configurations with fixed l^2-norm (power). These are spatially localized solutions that could be in principle studied experimentally, see [8, 10, 29] for some related localized solutions. The existence of the energy minimizers follows from a concentration-compactness argument that is similar to the one used in [20, 22, 30, 32].

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We also show that the minimizer is even and decays monotonically away from its maximum. These two properties follow from discrete symmetrization and rearrangement arguments, see Theorem 2.2 for statements and proofs. The notion of symmetrization for functions on the lattice uses the Fourier transform, while the monotonicity argument uses a discrete analogue of Riesz's rearrangement inequality, see [17].

We note that [1] shows the existence of spatially decaying breathers using different arguments, assuming that the linear intersite coupling parameter δ is small in absolute value. The present results assume instead that $\delta\gamma > 0$, see (2.1), and thus extend the existence results of [1] to a larger parameter range. The $\delta\gamma > 0$ case is also the one relevant to experiments.

The numerical results of [1] suggest that the linearization around the minimizing solutions of section 2 should have internal modes. Showing this in the general case seems difficult and we here present results for the linearization around one-peak breather solutions obtained for small linear intersite coupling δ via a continuation argument, see [1]. These solutions exist in the intersection of the parameter ranges considered in Section 2 and in [1], and we argue that they coincide with the energy minimizers of Section 2, see Proposition 4.3 for the precise statement. The presence of stable internal modes was seen explicitly in the limiting case on vanishing linear coupling δ in [1]. In Section 4 we use perturbation arguments, especially the Krein signature, to show the persistence of at least a finite number of stable internal modes for $\delta > 0$ sufficiently small, see Proposition 4.2. These results agree with the numerical observations of [1].

The second type of solutions we study are shelf-type breather solutions. These are solutions that decay in one direction and asymptote to a nonzero value in the other direction. These solutions have infinite l^2-norm, and their existence is shown by an implicit function theorem that allows us to continue solutions of the $\delta = 0$ equation, see [1, 18, 24, 27] for related results for solutions that decay at infinity. The argument applies to both the power and nonlocal nonlinearities, see Section 3. The power case is here primarily interesting as a limit of the nonlocal case. In the nonlocal case we continue from a nontrivial configuration that exhibits a maximum at the interface between sites with vanishing and nonvanishing amplitude. This effect was seen originally in [1] for l^2 solutions, and we see here that the linearization can be still inverted explicitly.

We also consider the linearization around the shelf-type breathers of Section 3 and present results on the essential spectrum of the corresponding operators. In the limiting case of the power nonlinearity, the essential spectrum is obtained by studying the "spatial" dynamics of the piecewise constant coefficient linear map defined by the linear operator. This is a common approach for problems on the line, see e.g. [14], and we show discrete analogues of some of the basic results in Subsection 4.2. In the nonlocal case we do not seem to have an immediate correspondence of the operator problem to spatial dynamics, but we can reproduce some of the calculations of the local case using the fact that the operator problem involves piecewise translation invariant (convolution) operators to obtain a weaker result for the essential spectrum, see Theorem 4.2 in Subsection 4.3. A possible alternative approach may be provided by a study of the analytic properties of the resolvent, see [12] for discrete Schrödinger operators, and [28] for the linearization around NLS breathers.

We note that the motivation for studying the shelf-type solutions is heuristic at present. First, the shelf-type breathers can be thought of as limits of a class of breathers that have finite support in the $\delta = 0$ limit. see [1]. These "finite shelf-type breather" solutions can be characterized by an integer $m > 0$, the size of their support in the $\delta = 0$ limit. The essential spectrum computed in Section 4 is the apparent limit of a subset of the spectrum of the linearization around the finite shelf-type breathers as m increases, see details in Section 4. We also point out that there are interesting connections between shelf-type breathers of discrete NLS equations and static front solutions of discrete nonlinear diffusion equations. The equations for these solutions are similar, and the continuation methods of Section 3 can be also used to show existence and monotonicity properties of static fronts, see [4, 11]. In Section 3 we remark on the discrete version of a recently proposed diffusion equation with nonlocal nonlinearity, see [9]. Also, in discrete diffusion equations the static solutions can become "depinned", i.e. may start to move after some value of the intersite coupling, and the static problem is a starting point for analyzing these moving fronts [2]. In the discrete NLS we do not have a clear analogue of the depinning transition for shelf-type solutions, but a preliminary numerical study [21] suggests that such a transition may be possible. This transition can be related to changes in the spectrum and bifurcations of the shelf-type breathers as we increase the linear coupling.

The Fratalocchi–Assanto model can be generalized to higher dimensions, see e.g. [31, 32] for dimension dependent phenomena in the local system, however the experimental setup of [7, 8, 10, 29] does not have a higher dimensional analogue. Also, comparisons of the model to experiments have used δ, γ, κ near unity, see [7], and thus the results of Section 2 are directly relevant. The small δ results of Sections 3, 4 agree with the numerical results of [1], where δ was up to 0.5. Their relevance to the experimental parameter range requires further numerical study.

The paper is organized as follows. In Section 2 we study energy minimizing breathers and show existence, and symmetry and monotonicity properties. In Section 3 we show the existence of shelf-like solutions for the local and nonlocal nonlinearities, in Subsections 3.1, 3.2 respectively. In Section 4 we study the spectra of the linearization around different breathers. In Subsection 4.1 we study the linear stability of one-peak breather solutions in the small linear coupling regime and relate these solutions to the minimizers of Section 2. In Subsections 4.2, 4.3 we present results on the essential spectra of shelf-type solutions in the local and nonlocal cases, respectively.

2. Nonlocal discrete NLS and energy-minimizing breather solutions

We consider the one-dimensional discrete NLS equation

$$\dot{u}_n = \delta i(u_{n+1} + u_{n-1} - 2u_n) + 2\gamma \tanh\frac{\kappa}{2} i\Big(\sum_{m\in\mathbb{Z}} e^{-\kappa|m-n|}|u_m|^2\Big)u_n \qquad n\in\mathbb{Z}, \tag{2.1}$$

with δ, γ, κ real constants, $\kappa > 0$, see [7].

Equation (2.1) can be written formally as the Hamiltonian system

$$\dot{u}_n = -i\frac{\partial H}{\partial u_n^*}, \quad n\in\mathbb{Z}, \quad \text{with} \tag{2.2}$$

$$H = \delta\sum_{n\in\mathbb{Z}}|u_{n+1} - u_n|^2 - \gamma\tanh\frac{\kappa}{2}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|u_m|^2 e^{-\kappa|m-n|}|u_n|^2. \tag{2.3}$$

The quantity $P = \sum_{n\in\mathbb{Z}}|u_n|^2$ is formally a conserved quantity.

We consider solutions of equation (2.1) of the "breather" form $u_n = e^{-i\omega t}A_n$, with ω real, and $A:\mathbb{Z}\to\mathbb{C}$ independent of t. Such A satisfies

$$-\omega A_n = \delta(A_{n+1} + A_{n-1} - 2A_n) + 2\gamma\tanh\frac{\kappa}{2}\Big(\sum_{m\in\mathbb{Z}} e^{-\kappa|m-n|}|A_m|^2\Big)A_n, \qquad \forall n\in\mathbb{Z}. \tag{2.4}$$

Let $||\cdot||_p$ denote the l^p-norm for complex-valued functions on $\mathbb{Z}$. Let $Y = l^2(\mathbb{Z},\mathbb{C})$ with the norm $||\cdot|| = ||\cdot||_2$.

Let $\{\hat{e}_n\}_{n\in\mathbb{Z}}$ denote the standard basis in Y.

Let $\lambda > 0$ and define

$$I_\lambda = \inf\{H(v) : v \in Y), P(v) = \lambda\}. \tag{2.5}$$

A configuration $A_* \in Y$, $P(A_*) = \lambda$, satisfying $H(A_*) = I_\lambda$ is referred to as a *minimizer* or ground state (of H at power $P = \lambda$). The set of minimizers of H at power λ is denoted as $\mathcal{M}_\lambda$. For $q \in \mathbb{Z}$, $\tau_q A$ denotes the translation of A by q, i.e. $(\tau_q A)_n = A_{n-q}$. The operation of multiplying $A \in Y$ by a scalar $e^{i\phi}$, $\phi \in \mathbb{R}$, is referred to a global phase change. Translations and global phase changes leave P and H invariant.

THEOREM 2.1. *Let δ, $\gamma > 0$. Then for every $\lambda > 0$, the set $\mathcal{M}_\lambda$ is nonempty, and $A \in \mathcal{M}_\lambda$ implies that A satisfies property (2.4).*

Proof. We will only present a sketch of the proof. The existence part uses a discrete version of Lyon's concentration-compactness principle, following [20, 22, 30, 32]. The functional H is easily seen to be bounded below, and has the form $H = \delta H_2 - \gamma V$, with H_2, V homogeneous quadratic and quartic functionals respectively. Both $H_2(u)$ and $V(u)$ are positive for any $u \neq 0$ in X. The discrete version of the concentration-compactness lemma (see [22], Sections 3, 4) states that there are three scenarios: splitting, vanishing, and convergence (up to translation), indicated by the number Γ in [22], (3.13). The goal is to show convergence, for any $\lambda > 0$. To eliminate splitting, we use the subadditivity property (Lemma 4.3, [22]), and a discrete analogue of Lemma 4.4 of [23]. These arguments are valid for all $\lambda > 0$. To eliminate vanishing for a given $\lambda > 0$ it suffices to find a test function $f \in X$, with $P(f) = \lambda$, and $H(f) < 0$. We use functions

$$f = Cg, \quad g \in X, \quad C = \sqrt{\lambda}||g||^{-1}. \tag{2.6}$$

Then $P(f) = \lambda$, and the condition $H(f) < 0$ becomes

$$\frac{\delta ||Dg||^2 ||g||^2}{\gamma V(g)} < \lambda, \tag{2.7}$$

where $(Dg)_n = g_{n+1} - g_n$, $n \in \mathbb{Z}$. We use test functions $(g)_n = e^{-\alpha|n|}$, $n \in \mathbb{Z}$, with $\alpha > 0$ (as in [32]) to see that

$$||Dg||^2 ||g||^2 = 2 + O(\alpha), \quad V(g) \geq ||g||_4^4 = \frac{1}{2\alpha} + O(1), \quad \text{as} \quad \alpha \to 0^+. \tag{2.8}$$

Then, given any $\lambda > 0$, the left hand side of inequality (2.7) can be made smaller by taking $\alpha > 0$ sufficiently small. It follows that a minimizing sequence will converge up to translations and global phase rotations, and that $\mathcal{M}_\lambda$ is nonempty for all $\lambda > 0$.

We easily check that $H : Y \to \mathbb{R}$ is C^1, and that $H'(A) - \omega P'(A) = 0$ implies that A is a solution of equation (2.4). ∎

THEOREM 2.2. *Let δ, $\gamma > 0$, and consider $\lambda > 0$. Then $A \in \mathcal{M}_\lambda$ implies that there exist $\phi \in \mathbb{R}$, $q \in \mathbb{Z}$ such that $A = e^{i\phi}\tau_q \overline{A}$, with $\overline{A}$ real and positive, reflection symmetric and non-increasing, i.e. $\overline{A}_n \geq 0$, $\forall n \in \mathbb{Z}$, $\overline{A}_{-n} = \overline{A}_n$, $\forall n \in \mathbb{Z}$, and $\overline{A}_a \geq \overline{A}_b$, for all positive integers $a < b$.*

We first define a symmetrization operation that transforms any $u \in Y = l^2(\mathbb{Z},\mathbb{C})$ into a configuration $w \in Y$ that is real, non-negative and symmetric with respect to the origin, and that moreover has the same l^2-norm and a smaller value of H. The symmetrization, and the proof of Lemma 2.1 below were suggested to us by Diego Rial.

For $f \in Y$, define $\tilde{f}: \mathbb{R} \to \mathbb{C}$ by

$$\tilde{f}(x) = \sum_{n \in \mathbb{Z}} f_n e^{inx}, \quad x \in \mathbb{R}. \tag{2.9}$$

$\tilde{f}$ is clearly $2\pi-$periodic. By the L^2 theory for Fourier series, $\tilde{f}$ belongs to $L^2(S^1, \mathbb{C})$, with S^1 the interval $[0, 2\pi]$ with 0, 2π identified. Moreover equation (2.9) defines an isomorphism between Y and $L^2(S^1, \mathbb{C})$. The inverse is given by

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(x) e^{-inx}\, dx, \quad n \in \mathbb{Z}, \tag{2.10}$$

i.e. $\tilde{f}$ is the inverse Fourier transform of f. Moreover, we have Parseval's identity

$$\sum_{n \in \mathbb{Z}} |f_n|^2 = \frac{1}{2\pi} \int_{S^1} |\tilde{f}(x)|^2\, dx. \tag{2.11}$$

Consider $u \in Y$, with $\tilde{u}$ as in equation (2.9), and define v, w as follows:

$$v_n = \frac{1}{2\pi} \int_{S^1} |\tilde{u}(x)|\, e^{-inx}\, dx, \quad n \in \mathbb{Z}, \tag{2.12}$$

and

$$w_n = |v_n|, \quad n \in \mathbb{Z}. \tag{2.13}$$

It follows from identity (2.11) that v, w belong to Y, and that $||w|| = ||v|| = ||u||$.

Also, by definition (2.12) we have $v_{-n} = v_n^*$, $\forall n \in \mathbb{Z}$. Then definition (2.13) implies that

$$w_{-n} = w_n \geq 0, \quad \forall n \in \mathbb{Z}. \tag{2.14}$$

The configuration w is therefore real, positive, and symmetric, with $||w|| = ||u||$.

LEMMA 2.1. *Consider $u \in Y$, and v, $w \in Y$ as in definitions (2.12) and (2.13). Then, $H(w) \leq H(v) \leq H(u)$.*

Proof. First we examine the quartic part V of H in Fourier space. Letting $g_n = e^{-\kappa|n|}$, $n \in \mathbb{Z}$, we compute

$$\tilde{g}(x) = \sum_{n \in \mathbb{Z}} e^{-\kappa|n|} e^{inx} = \frac{1 - e^{-2\kappa}}{1 - 2e^{-\kappa}\cos x + e^{-2\kappa}}. \tag{2.15}$$

We observe that $|\tilde{g}(x)| = \tilde{g}(x)$, $\forall x \in \mathbb{R}$.

Also,

$$V(u) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |u_n|^2 g_{n-m} |u_m|^2 \tag{2.16}$$

$$= \frac{1}{8\pi^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \tilde{g}(x)\, \tilde{u}(y)\, \tilde{u}^*(y+x)\, \tilde{u}(z)\, \tilde{u}^*(z-x)\, dx\, dy\, dz \tag{2.17}$$

$$\leq \frac{1}{8\pi^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\tilde{g}(x)\,|\tilde{u}(y)|\,|\tilde{u}^*(y+x)|\,|\tilde{u}(z)|\,|\tilde{u}^*(z-x)|\,dx\,dy\,dz. \quad (2.18)$$

The definition of v in equation (2.12) and the isomorphism between Y and $L^2(S^1,\mathbb{C})$ imply $|\tilde{u}(x)|=\tilde{v}(x)$, a.e. $x\in S^1$, so that the R.H.S. of inequality (2.18) is $V(v)$. Therefore

$$V(u)\leq V(v). \quad (2.19)$$

On the other hand, by the definition of w in equation (2.13) we also have $V(v)=V(w)$, hence

$$V(u)\leq V(w). \quad (2.20)$$

For the quadratic part H_2 we calculate

$$H_2(u)=\frac{1}{\pi}\int_0^{2\pi}\sin^2\frac{x}{2}\,|\tilde{u}(x)|^2\,dx=H_2(v), \quad (2.21)$$

since $|\tilde{u}(x)|=\tilde{v}(x)$, a.e. $x\in S^1$. Then

$$H_2(v)=\sum_{n\in\mathbb{R}}|v_{n+1}-v_n|^2\geq\sum_{n\in\mathbb{R}}\left||v_{n+1}|-|v_n|\right|^2=H_2(w), \quad (2.22)$$

using the definition of w in equation (2.13). Combining inequalities (2.20) and (2.22) we conclude that $H(u)\geq H(w)$, as stated. □

We also observe that by equation (2.12)

$$v_0=\frac{1}{2\pi}\int_0^{2\pi}|\tilde{u}(x)|dx\geq 0;\quad |v_n|\leq v_0,\quad \forall n\in\mathbb{Z}\setminus\{0\}. \quad (2.23)$$

Then equation (2.13) implies

$$w_n\leq w_0,\quad \forall n\in\mathbb{Z}\setminus\{0\}, \quad (2.24)$$

i.e. the maximum of w is attained at the origin.

The second operation that decreases H while keeping the l^2-norm fixed is that of replacing $u\in Y$ by its non-increasing rearrangement. To define this operation, let $\mathcal{S}$ be the set of all $f:\mathbb{Z}\to\mathbb{R}$ that satisfy (i) $f_{-n}=f_n$, $\forall n>0$, (ii) $f_0>0$, (iii)$f_0\geq f_n\geq 0$, $\forall n>0$, and (iv) $f_n\to 0$ as $|n|\to\infty$.

Note that $\mathcal{S}$ is the union of its disjoint subsets $\mathcal{S}^+$, $\mathcal{S}^0$, and $\mathcal{S}^F$, where $f\in\mathcal{S}^+$ implies $f_n>0$, $\forall n\in\mathbb{Z}$, $f\in\mathcal{S}^0$ implies that there exist n for which $f_n=0$ and that $f_n\neq 0$ for an infinite subset of $n\in\mathbb{Z}$, and $f\in\mathcal{S}^F$ implies that the set of n for which $f_n\neq 0$ is finite.

For $f\in\mathcal{S}^+\cup\mathcal{S}^0$ as above, let f^j the j-th largest value of f, starting from $j=1$, i.e. $f^1=f_0$, and $j\in\mathbb{Z}^+=\{1,2,\dots\}$, for all $f\in\mathcal{S}$. For if $f\in\mathcal{S}^F$, then f attains a finite number $\mathcal{T}(f)$ of values; then f^j, $j\leq\mathcal{T}(f)$ is the j-th largest value of f, starting from $j=1$, and we set $f^j=0$, for all $j>\mathcal{T}(f)$.

Given a subset J of $\mathbb{Z}$, χ_J will denote the characteristic function of J. We also use the abbreviated notation $\chi_{f\geq a}$ to denote the characteristic function of the set of $n\in\mathbb{Z}$ satisfying $f_n\geq a$. Note that the support of $\chi_{f\geq f^j}$ is finite, for any $f\in\mathcal{S}$, and $j\geq 1$. In what follows $\chi_J(n)$ denotes the value of χ_J at site n.

Definition 2.1. *Let $\mathcal{I}^S$ be the set of finite $I\subset\mathbb{Z}$ satisfying $0\in I$, and $n\in I$ implies $-n\in I$. Let m be such that $|I|=2m+1$. Then $\overline{I}$ denotes the set of $2m+1$ consecutive integers from $-m$ to m.*

Let $f\in S^+\cup S^F$, fix $F^j>0$ and let $I=\{n\in\mathbb{Z}: f_n\geq f^j\}=\mathrm{supp}\chi_{f\geq f^j}$. Then $I\in\mathcal{I}^S$ and we let $\overline{\chi}_{f\geq f^j}=\chi_{\overline{I}}$.

For $f\in\mathcal{S}$ we define its *symmetric non-increasing rearrangement* as the function $\overline{f}:\mathbb{Z}\to\mathbb{R}^+\cup\{0\}$ defined by

$$\overline{f}=\sum_{j=1}^{\infty}(f^j-f^{j+1})\overline{\chi}_{f\geq f^j}. \tag{2.25}$$

We check that $\overline{f}\in\mathcal{S}^+\cup\mathcal{S}^F$, and that $\overline{f}_m\leq\overline{f}_n$ for all $m>n\geq 0$. We recall the following additional facts.

Proposition 2.1. *If $f\in\mathcal{S}$ then*

$$g_n=\sum_{j=1}^{\infty}(f^j-f^{j+1})\chi_{f\geq f^j}(n),\quad n\in\mathbb{Z}, \tag{2.26}$$

coincides with f, i.e. $g_n=f_n$, $\forall n\in\mathbb{Z}$. If also $f\in\mathcal{S}\cap l^1$, then

$$\sum_{n\in\mathbb{Z}}\overline{f}_n=\sum_{n\in\mathbb{Z}}f_n. \tag{2.27}$$

Remark 2.1. The second conclusion of Proposition 2.1 also implies that $||\overline{h}||_{l^p}=||h||_{l^p}$, for all $h\in\mathcal{S}\cap l^p$, $p>0$. This follows by setting $f_n=(h_n)^p$.

Proof. First, suppose that n is such that $f_n=f^m$, with $f^j>0$. Then $\chi_{f\geq f^j}(n)=1$, if $j\geq m$, and vanishes otherwise, and therefore

$$g_n=\sum_{j=m}^{\infty}(f^j-f^{j+1})=\lim_{N\to\infty}(f^m-f^N)=f^m=f_n, \tag{2.28}$$

since $f^j\to 0$ as $j\to\infty$. In the case where $f_n=0$ for some n, then $g_n=0$ by equation (2.25), since n is not in the support of any $\chi_{f\geq f^j}$ with $f^j>0$.

For the second statement, we first consider the case $f\in\mathcal{S}^+$. Let $\mathcal{N}(f^j)$, $j\in\mathbb{Z}^+$ be the set of n satisfying $f_n=f^j$. The $\mathcal{N}(f^j)$ are finite, mutually disjoint, and their union is $\mathbb{Z}$. Moreover the set of f^j, $j\in\mathbb{Z}^+$ is the set of values of f. Then

$$\sum_{n\in\mathbb{Z}}f_n=\sum_{j=1}^{\infty}f^j\,|\mathcal{N}(f^j)|, \tag{2.29}$$

moreover $\mathcal{N}(f^j)=\mathrm{supp}\chi_{f\geq f^j}$. Therefore equation (2.29) implies

$$\sum_{n\in\mathbb{Z}}f_n=\sum_{j=1}^{\infty}f^j\,|\chi_{f\geq f^j}|=\sum_{j=1}^{\infty}f^j\,|\overline{\chi}_{f\geq f^j}|, \tag{2.30}$$

since the support of $\chi_{f\geq f^j}$ belongs to $\mathcal{I}^S$. We check that $\overline{f}^j=f^j$, and that $\overline{\chi}_{f\geq f^j}=\chi_{\overline{f}\geq\overline{f}^j}$, $\forall j\in\mathbb{Z}^+$. Thus equation (2.30) implies

$$\sum_{n\in\mathbb{Z}}f_n=\sum_{j=1}^{\infty}\overline{f}^j\,|\chi_{\overline{f}\geq\overline{f}^j}|=\sum_{j=1}^{\infty}\overline{f}^j\,\mathcal{N}(\overline{f}^j)=\sum_{n\in\mathbb{Z}}\overline{f}_n. \tag{2.31}$$

The case $f \in \mathcal{S}^0$ is similar. We note there that the set where $f_n = 0$ is not included in the sum in equation (2.29), i.e. we only sum over the positive values of f. The set of positive values of f is precisely the set of the f^j, $j \in \mathbb{Z}^+$, and the remaining steps are identical since the sets $\mathcal{N}(f^j)$ satisfy the same properties. In the case $f \in \mathcal{S}^F$ we only sum over the values f^j, $j \leq \mathcal{T}(f)$ in equation (2.29). The remaining steps are identical as the $\mathcal{N}(f^j)$, $1 \leq j \leq \mathcal{T}(f)$, satisfy the same properties. ∎

From the properties of the configuration $w \in Y$ obtained from $u \in Y$ via equations (2.12) and (2.13), as above, it is clear that $w \in \mathcal{S} \cap l^2$, and we can apply the operation of non-increasing rearrangement to obtain a new symmetric configuration $\overline{w}$ that is non-increasing, and satisfies $||\overline{w}|| = ||w||$. To show that $H(\overline{w}) \leq H(w)$, we use Lemma 2.2 below on rearrangements of quadratic forms. The lemma is a special case of a discrete analogue of the Riesz rearrangement inequality, see e.g. [17].

Let k be a positive integer and let I, J be finite subsets of $\mathbb{Z}$, and let

$$Q(I,k,J) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \chi_I(n) \mathcal{H}_k(m-n) \chi_J(m), \tag{2.32}$$

where $\mathcal{H}_k(n) = 1$ if $|n| \leq k$, and vanishes otherwise.

LEMMA 2.2. *Let I, $J \in \mathcal{I}^S$, and consider the corresponding $\overline{I}$, $\overline{J}$, as in Definition 2.1. Assume also that $I \subseteq J$. Then for any positive integer k*

$$Q(\overline{I},k,\overline{J}) \geq Q(I,k,J). \tag{2.33}$$

Given a pair I, J the inequality is strict for at least one positive k, unless $\overline{I} = I$, $\overline{J} = J$.

Proof. Fix a positive integer k. Consider first the case $I = J$. Let $1 \leq j_1 < j_2, \ldots, < j_m$ denote the positive sites of J. Then $|J| = 2m+1$. A *gap* G of J is a symmetric set of consecutive sites j not in J that satisfy either $j_\lambda < j < j_{\lambda+1}$ or $-j_{\lambda+1} < j < -j_\lambda$ for a pair of $0 < j_\lambda < j_{\lambda+1}$ in J. Note that the definition implies that $g = j_{\lambda+1} - j_\lambda > 1$, and that G consists of $2g > 0$ sites. A gap is specified by the pair of sites j_λ, $j_{\lambda+1}$ above, referred to below as its (positive) *edges*.

If $J = \overline{J}$, then J has no gaps. The idea is to show that deleting gaps in I and J simultaneously increases Q. We then generalize to the case $I \subset J$.

Given a gap G specified by the sites j_λ, $j_{\lambda+1}$ of J as above, let $J\text{sub}G \in \mathcal{I}^S$ be the set whose positive elements are all $j \in J$ that satisfy $j \leq j_\lambda$, and all $j - g$, $j \in J$ that satisfy $j \geq j_\lambda + g + 1$. It is easy to show that $Q(J\text{sub}G, k, J\text{sub}G) \geq Q(J,k,J)$. Iterating the elimination of gaps we obtain $Q(\overline{J},k,\overline{J}) \geq Q(J,k,J)$.

In the case $I \subseteq J$, the definition of gaps is the same. We first eliminate all gaps of J whose positive sites are to the right of the largest site of I. The resulting set is denoted as J_1, and we easily argue that $Q(I,k,J_1) \geq Q(I,k,J)$. Also $I \subseteq J_1$, moreover the remaining gaps of J_1 are also gaps of I, otherwise there would be sites of I not in J_1. Eliminating all the gaps of J_1 from both I and J_1 we obtain new sets I_2, and J_2 respectively. We have $J_2 = \overline{J}$, and we easily show that $Q(I_2,k,\overline{J}) \geq Q(I_2,k,J_1)$. Also $I_2 \subseteq \overline{J}$. If $I_2 \neq \overline{J}$, then I_2 still has gaps, and we eliminate them to obtain $\overline{I}$. We again can show that $Q(\overline{I},k,\overline{J}) \geq Q(I_2,k,\overline{J})$. The above hold for any positive integer k.

It is easy to show in each of the above steps, that cancellation of any gap would give strict inequality for all values of $k \geq 1$. For instance, the relation $I \subseteq J$ at all stages implies that the elimination of the common points in the gap will increase for each i in the smaller set the number of j in the larger set satisfying $|i-j| \leq k$. The increase will be strict for the points at the edges of the gap, for all $k \geq 1$. ∎

LEMMA 2.3. *Consider the configuration $w \in \mathcal{S} \cap Y$, and its non-increasing symmetrization $\overline{w} \in Y$, as above. Then $H(\overline{w}) \leq H(w)$. The inequality is strict, unless $w = \overline{w}$.*

Proof. We first consider the quartic part V, using the functions $f_n = w_n^2$, $\overline{f}_n = \overline{w}_n^2$, $\forall n \in \mathbb{Z}$, i.e. $f, \overline{f} \in \mathcal{S} \cap l^1$. Also let $g(n) = e^{-\kappa|n|}$, $\forall n \in \mathbb{Z}$. We can use Proposition 2.1 and the definition $\mathcal{H}_k$ after equation (2.32) to write $g = \sum_{j=1}^{\infty}(g^j - g^{j+1})\mathcal{H}_j$. Then,

$$V(u) = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} f_n g(n-m) f_m \tag{2.34}$$

$$= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} [\sum_{j_1=1}^{\infty} a_{j_1} \chi_{f\geq f^{j_1}}(n)][\sum_{j=1}^{\infty} b_j \mathcal{H}_j(m-n)][\sum_{j_2=1}^{\infty} a_{j_2}\chi_{f\geq f^{j_2}}(m)], \tag{2.35}$$

with $a_j = f^j - f^{j+1}$, $b_j = g^j - g^{j+1}$.

The order of summations can be changed using the l^2-summability of w, and the fact that convolution with g defines a bounded operator in Y (we omit therefore the steps of defining all quantities for finite sums and taking limits).

We then have

$$V(u) = \sum_{j_1=1}^{\infty}\sum_{j=1}^{\infty}\sum_{j_2=1}^{\infty} a_{j_1} b_j a_{j_2} \left[\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \chi_{f\geq f_1^j}(n)\mathcal{H}_j(m-n)\chi_{f\geq f_2^j}(m)\right] \tag{2.36}$$

$$= \sum_{j_1=1}^{\infty}\sum_{j=1}^{\infty}\sum_{j_2=1}^{\infty} a_{j_1} b_j a_{j_2} Q(\mathrm{supp}\chi_{f\geq f^{j_1}}, j, \mathrm{supp}\chi_{f\geq f^{j_2}}). \tag{2.37}$$

Since $\mathrm{supp}\chi_{f\geq f^j} \subseteq \mathrm{supp}\chi_{f\geq f^{j'}}$ if $f^j \geq f^{j'}$, we can apply Lemma 2.2 for the supports of the $\chi_{f\geq f^j}$ to obtain from equation (2.37)

$$V(f) \geq \sum_{j_1=1}^{\infty}\sum_{j=1}^{\infty}\sum_{j_2=1}^{\infty} a_{j_1} b_j a_{j_2} Q(\mathrm{supp}\overline{\chi}_{f\geq f^{j_1}}, j, \mathrm{supp}\overline{\chi}_{f\geq f^{j_2}}) \tag{2.38}$$

$$= V(\overline{f}). \tag{2.39}$$

The inequality is strict unless

$$Q(\mathrm{supp}\chi_{f\geq f^{j_1}}, j, \mathrm{supp}\chi_{f\geq f^{j_2}}) = Q(\mathrm{supp}\overline{\chi}_{f\geq f^{j_1}}, j, \mathrm{supp}\overline{\chi}_{f\geq f^{j_2}}), \tag{2.40}$$

for all j_1, j_2. In that case we see that $\overline{w} = w$.

The quadratic part H_2 can be written as

$$H_2(w) = \sum_{n\in\mathbb{Z}}(w_{n+1}^2 + w_n^2 - 2w_n w_{n+1}) = 2||w||^2 - 2\sum_{n\in\mathbb{Z}} w_n w_{n+1}, \tag{2.41}$$

and we define H_2^+ by

$$H_2^+(w) = \sum_{n\in\mathbb{Z}}(w_{n+1}^2 + w_n^2 + 2w_n w_{n+1}) = 2||w||^2 + 2\sum_{n\in\mathbb{Z}} w_n w_{n+1}. \tag{2.42}$$

To show that $H_2(\overline{w}) \leq H_2(w)$ it then suffices to show that $H_2^+(\overline{w}) \geq H_2^+(w)$. We write

$$H_2^+(w) = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} w_n M(n-m) w_m, \tag{2.43}$$

with $M(0)=2$, $M(\pm 1)=1$, and $M(n)=0$ for $n\in\mathbb{Z}\setminus\{-1,0,1\}$. M is symmetric and decreasing in $\mathbb{Z}^+$, and we can then follow the argument used for V above to see that $H_2^+(\overline{w})\geq H_2^+(w)$, as required. It follows immediately by $H(\overline{w})=H_2(\overline{w})-V(\overline{w})\leq H_2(w)-V(w)=H(w)$, with strict inequality if $\overline{w}\neq w$, as stated. ∎

REMARK 2.2. In the case δ, $\gamma<0$ we have a similar statement for the maximizer of H, i.e. we apply the theorem to the minimizer of $-H$.

REMARK 2.3. For $\delta>0$, $\gamma<0$ and any $\lambda>0$ we have $I_\lambda=0$. We can see that by considering the functions g^N, $N\in\mathbb{Z}^+$, where $g_n^N=\sqrt{\lambda}/\sqrt{(2N+1)}$ if $-N\leq n\leq N$, and vanishes elsewhere. It follows that the infimum is not attained and that we can not use minimization to find critical points. In the case of δ sufficiently small critical points were obtained by different arguments, see [1].

REMARK 2.4. The symmetrization and rearrangement results above also apply to energy minimizing breathers of the cubic DNLS in $\mathbb{Z}$, where we minimize $H_q=\delta H_2-\gamma H_4$, $H_4(u)=||u||_{l^4}^4$, over configurations $u\in Y$ with $P(u)=\lambda>0$. The existence of a minimizer for arbitrary $\lambda>0$ is shown in [32]. Lemma 2.1 also holds for H_4, using the $k\to\infty$ limit $\tilde{g}(x)=1$, $\forall x$, in equation (2.15). Lemma 2.3 clearly applies to H_q as well.

3. Infinite shelf-type breathers

In this section we show the existence of infinite shelf-type breather solutions of equation (2.4) with $|\delta|\neq 0$ and small. We consider two cases depending on the interaction term in equation (2.1): local case and nonlocal case, corresponding to "$\kappa=\infty$", and $\kappa>0$ respectively. In Theorem 3.1 we show the continuation of shelf-type breathers of equation (2.4) with $\delta=0$ in the local case. Theorem 3.2 concerns the continuation of shelf-type solutions for arbitrary $\kappa>0$.

3.1. Local case. Let $\omega\in\mathbb{R}$. For $A\in l^\infty(\mathbb{Z},\mathbb{R})$, $\delta\in\mathbb{R}$ and $n\in\mathbb{Z}$ define $F_n: l^\infty(\mathbb{Z},\mathbb{R})\times\mathbb{R}\longrightarrow\mathbb{R}$ by

$$F_n(A,\delta)=\delta(\Delta A)_n+2\gamma A_n^3+\omega A_n, \tag{3.1}$$

where $(\Delta A)_n=A_{n-1}-2A_n+A_{n+1}$.

Let $\alpha>0$ and define $S_0=\{n_0+1,n_0+2,n_0+3,\ldots\}\subset\mathbb{Z}$, $S_A=\mathbb{Z}\setminus S_0$. We consider a nontrivial solution $A=\widetilde{A}$ of $F_n(A,0)=0$, $\forall n\in\mathbb{Z}$, obtained by requiring $A_n\neq 0$ for $n\in S_A$, and $A_n=0$ for $n\in S_0$. As $\delta=0$, the equation $F_n=0$ becomes

$$-2\gamma|A_n|^2=\omega,\quad \forall n\in S_A. \tag{3.2}$$

Then γ and ω must have opposite signs and we consider the solution

$$\widetilde{A}_n=\begin{cases}0, \text{ if } n\in S_0,\\ \alpha, \text{ if } n\in S_A,\end{cases} \tag{3.3}$$

with $\alpha=\sqrt{-\frac{\omega}{2\gamma}}$.

REMARK 3.1. We can obtain more nontrivial solutions by taking $A_n<0$ for all $n\in S_A$. Moreover, we can choose $A_n\neq 0$ for $n\in S_A:=S_+\cup S_-$, with $A_n>0$ for $n\in S_+$ and $A_n<0$ for $n\in S_-$ provided S_+ is finite or S_- is finite. The arguments below apply to these solutions as well.

Let $X=l^2(\mathbb{Z},\mathbb{R})$ with the norm $\|\cdot\|=\|\cdot\|_X$ as in Section 2. Let $||\cdot||_{Z,Y}$ denote the operator norm of a linear operator from Z to Y, both Banach spaces.

We define $G: X\times\mathbb{R}\to X$ as

$$G(B,\delta)=\{G_n(B,\delta)\}_{n\in\mathbb{Z}}=F_n(B+\widetilde{A},\delta), \tag{3.4}$$

with F_n given by equation (3.1) It can be checked that G is well-defined, i.e. if $B\in X$, $\delta\in\mathbb{R}$, then $G(B,\delta)\in X$.

THEOREM 3.1. *Let G be as in definition (3.4) and consider the solution $(x_0,y_0)=(0,0)$ of $G(B,\delta)=0$. Then there exists $\epsilon_0>0$ and a unique continuous function $B:(-\epsilon_0,\epsilon_0)\to X$ satisfying $B(0)=0$ and $G(B(\delta),\delta)=0$ for all $\delta\in(-\epsilon_0,\epsilon_0)$.*

The proof uses the implicit function theorem and is similar to that of Theorem 3.2 below (with simpler computations) and is omitted.

3.2. Nonlocal case. We now show existence results for arbitrary $\kappa>0$ and $|\delta|$ small enough. Let $\omega\in\mathbb{R}$ and $\kappa>0$. For $A\in l^\infty(\mathbb{Z},\mathbb{R})$ and $\delta\in\mathbb{R}$ define F_n, $n\in\mathbb{Z}$, by

$$F_n(A,\delta)=\delta(\Delta A_n)+2\gamma\tanh\frac{\kappa}{2}\left(\sum_{m\in\mathbb{Z}}e^{-\kappa|m-n|}|A_m|^2\right)A_n+\omega A_n. \tag{3.5}$$

We consider, as before, a nontrivial solution $A=\widetilde{A}$ of $F_n(A,0)=0$, $\forall n\in\mathbb{Z}$, determined by $\alpha>0$, and sets $S_0=\{n_0+1,n_0+2,n_0+3,\ldots\}\subset\mathbb{Z}$, $S_A=\mathbb{Z}\setminus S_0$, for which $A_n>0$ if $n\in S_A$, and $A_n=0$ if $n\in S_0$. Evaluating $\delta=0$ in equation (3.5), the equation $F_n=0$ becomes

$$-2\gamma\tanh\frac{\kappa}{2}\sum_{m\in S_A}e^{-\kappa|m-n|}|A_m|^2=\omega,\qquad \forall n\in S_A. \tag{3.6}$$

We also write equation (3.6) as

$$MJ=\left(-2\gamma\tanh\frac{\kappa}{2}\right)^{-1}\omega\varepsilon, \tag{3.7}$$

where $J_m=A_m^2$, $\varepsilon=[\ldots,1,1,1]^T\in\mathbb{R}^{|S_A|}$ and M is defined implicitly by the previous equation. Letting $\rho=e^{-\kappa}$ we have M and its inverse M^{-1} explicitly as

$$M=\begin{bmatrix}\ldots & & & & \ldots\\ & 1 & \rho & \rho^2 & \rho^3\\ & \rho & 1 & \rho & \rho^2\\ & \rho^2 & \rho & 1 & \rho\\ \ldots & \rho^3 & \rho^2 & \rho & 1\end{bmatrix},\quad M^{-1}=\tfrac{1}{1-\rho^2}\begin{bmatrix}\ldots & & & & \ldots\\ & 1+\rho^2 & -\rho & 0 & 0\\ & -\rho & 1+\rho^2 & -\rho & 0\\ & 0 & -\rho & 1+\rho^2 & -\rho\\ \ldots & 0 & 0 & -\rho & 1\end{bmatrix}. \tag{3.8}$$

The calculation of M^{-1} used the fact that M is Toeplitz. It is clear from equation (3.8) that M^{-1} is also bounded in X. Hence the vector J is computed explicitly as

$$J=\left(-2\gamma\tanh\frac{\kappa}{2}\right)^{-1}\omega M^{-1}\varepsilon,\quad \text{where}\quad M^{-1}\varepsilon=\frac{1-\rho}{1+\rho}\left[\ldots,1,1,1,\frac{1}{1-\rho}\right]^T.$$

Letting $\alpha^2=\left(-2\gamma\tanh\frac{\kappa}{2}\right)^{-1}\omega\frac{1-\rho}{1+\rho}$, the solution $\widetilde{A}$ determined by $\alpha>0$ is

$$\widetilde{A}_n=\begin{cases}\alpha, & \text{if } n\in S_A,\ n<n_0\\ \frac{\alpha}{\sqrt{1-\rho}}, & \text{if } n\in S_A,\ n=n_0,\\ 0, & \text{if } n\in S_0.\end{cases} \tag{3.9}$$

In a similar way we can find nontrivial solutions such that $A_n<0$ for all $n\in S_A$. Moreover, as in Remark 3.1, we can find nontrivial solutions such that $A_n\neq 0$ for $n\in S_A:=S_+\cup S_-$, with $A_n>0$ for $n\in S_+$ and $A_n<0$ for $n\in S_-$ provided S_+ is finite or S_- is finite. The arguments below are valid for these solutions as well.

Then we define $G: X\times\mathbb{R}\to X$ as

$$G(B,\delta)=\{G_n(B,\delta)\}_{n\in\mathbb{Z}}=\{F_n(B+\widetilde{A},\delta)\}_{n\in\mathbb{Z}}. \tag{3.10}$$

The fact that G is well-defined follows from the arguments in the proof of Lemma 3.1 below.

Theorem 3.2. *Let G be as in equation (3.10) and fix ω, $\kappa>0$. Let n_0 be a positive integer, and consider a nontrivial solution $(\widetilde{A},0)$ of $G(B,\delta)=0$ with $S_A=\{\ldots,n_0-2,n_0-1,n_0\}$. Then there exists $\epsilon_0>0$ and a unique continuous function $B:(-\epsilon_0,\epsilon_0)\to X$ satisfying $B(0)=0$ and $G(B(\delta),\delta)=0$ for all $\delta\in(-\epsilon_0,\epsilon_0)$.*

Proof. As in the previous theorem, we apply the implicit function theorem around a solution $(\widetilde{A},0)$ of $G(B,\delta)=0$. We present here the (Fréchet) derivative D_1G and we show that it is a linear isomorphism of X. The continuity of G and D_1G at $(0,0)$ is proved in Lemmas 3.1 and 3.2 below. We have

$$D_1G(0,0)=\begin{pmatrix} M_1 & 0\\ 0 & M_2\end{pmatrix}, \tag{3.11}$$

with

$$M_1=2c\alpha^2\begin{bmatrix} \cdots & & & & \cdots\\ & 1 & \rho & \rho^2 & \frac{\rho^3}{\sqrt{1-\rho}}\\ & \rho & 1 & \rho & \frac{\rho^2}{\sqrt{1-\rho}}\\ & \rho^2 & \rho & 1 & \frac{\rho}{\sqrt{1-\rho}}\\ \cdots & \frac{\rho^3}{\sqrt{1-\rho}} & \frac{\rho^2}{\sqrt{1-\rho}} & \frac{\rho}{\sqrt{1-\rho}} & \frac{1}{1-\rho}\end{bmatrix},\quad M_2=\omega\begin{bmatrix} 1-\rho & 0 & 0 & \cdots\\ 0 & 1-\rho^2 & 0 & \\ 0 & 0 & 1-\rho^3 & \\ \cdots & & & \cdots\end{bmatrix},$$

and $\rho=e^{-\kappa}$, $c=2\gamma\tanh\frac{\kappa}{2}$.

As M_2 is an infinite diagonal matrix with $1-\rho\le M_2(n,n)<1$, $\forall n\in\mathbb{Z}$, to prove that D_1G is a linear isomorphism in X it is enough to show that M_1 is invertible. We see that

$$M_1^{-1}=\frac{1}{(c\alpha^2)(1-\rho^2)}\begin{bmatrix} \cdots & & & & \cdots\\ & 1+\rho^2 & -\rho & 0 & 0\\ & -\rho & 1+\rho^2 & -\rho & 0\\ & 0 & -\rho & 1+\rho^2 & \frac{-\rho}{\sqrt{1-\rho}}\\ \cdots & 0 & 0 & \frac{-\rho}{\sqrt{1-\rho}} & 1-\rho\end{bmatrix},$$

with M_1^{-1} clearly bounded in X. ∎

Remark 3.2. Observe that the nonlocal case approaches the local case as $\kappa\to\infty$, and $\rho=e^{-\kappa}\to 0$.

Remark 3.3. We note that [9] propose a discrete nonlinear diffusion equation whose static front solutions satisfy equation (3.6) with the term $(1-\omega)A_n+A_n^2$ added to the right hand side (one may assume $A_n>0$ for all $n\in S_A$). The numerical results of [9] suggest that this variant of equation (3.6) also has a solution with a maximum at the interface.

To prove the continuity of G and D_1G at $(0,0)$ we will need to use the following discrete version of Young's inequality:

THEOREM 3.3 (Young's inequality, [17], ch. 4). *Let p, q, $r\geq 1$, $\frac{1}{p}+\frac{1}{q}>1$ and $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. Then $u\in l^p$, and $v\in l^q$ imply $u*v\in l^r$ and*

$$\|u*v\|_r\leq \|u\|_p\|v\|_q. \tag{3.12}$$

The inequality follows from the statement for functions on the line, see e.g. [17], ch. 4.

LEMMA 3.1. *Let G be as in equation (3.10) and fix ω, and $\kappa>0$. Then G is continuous at $(0,0)\in X\times\mathbb{R}$.*

Proof. Let $(B,\delta)\in X\times\mathbb{R}$, then

$$\begin{aligned}\|G(B,\delta)-G(0,0)\|_X^2&=\sum_{n\in\mathbb{Z}}|F_n(B+\tilde{A},\delta)-F_n(\tilde{A},0)|^2\\&=\sum_{n\in\mathbb{Z}}\Bigg|\delta\Delta(B_n+\widetilde{A}_n)+c_{\gamma,\kappa}(B_n+\widetilde{A}_n)\sum_{m\in\mathbb{Z}}e^{-\kappa|m-n|}|B_m+\widetilde{A}_m|^2\\&\quad+\omega(B_n+\widetilde{A}_n)-c_{\gamma,\kappa}\left(\sum_{m\in\mathbb{Z}}e^{-\kappa|m-n|}\widetilde{A}_m^2\right)\widetilde{A}_n-\omega\widetilde{A}_n\Bigg|^2,\end{aligned}\tag{3.13}$$

with $c_{\gamma,\kappa}=2\gamma\tanh\frac{\kappa}{2}$, and

$$C_n=\sum_{m\in\mathbb{Z}}e^{-\kappa|m-n|}\left(|B_m+\widetilde{A}_m|^2(B_n+\widetilde{A}_n)-\widetilde{A}_m^2\widetilde{A}_n\right),\quad \forall n\in\mathbb{Z}. \tag{3.14}$$

Then

$$\|G(B,\delta)-G(0,0)\|_X^2\leq 6\left(|\delta|^2\|\Delta\|_{X,X}^2\|B\|_X^2+|\omega|\,\|B\|_X^2+2|\gamma|\tanh\frac{\kappa}{2}\|C\|_X^2\right). \tag{3.15}$$

The first and second term in the last inequality vanish as $\delta\to 0$ and $B\to 0\in X$. It is then enough to show that $\|C\|_X$ vanishes as $\delta\to 0$ and $B\to 0\in X$. We have

$$\|C\|_X^2=\sum_{n\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}}\rho^{|m-n|}\left(|B_m+\widetilde{A}_m|^2(B_n+\widetilde{A}_n)-\widetilde{A}_m^2\widetilde{A}_n\right)\right|^2 \tag{3.16}$$

$$=\sum_{n\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}}\rho^{|m-n|}|B_m+\widetilde{A}_m|^2B_n+\sum_{m\in\mathbb{Z}}\rho^{|m-n|}\left(|B_m+\widetilde{A}_m|^2-\widetilde{A}_m^2\right)\widetilde{A}_n\right|^2 \tag{3.17}$$

$$\leq\sum_{n\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}}\rho^{|m-n|}|B_m+\widetilde{A}_m|^2B_n+\sum_{m\in\mathbb{Z}}\rho^{|m-n|}B_m(B_m+2\widetilde{A}_m)\widetilde{A}_n\right|^2 \tag{3.18}$$

$$\leq\Gamma(||B||,||A||_\infty)\sum_{n\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}}\rho^{|m-n|}B_n+\sum_{m\in\mathbb{Z}}\rho^{|m-n|}B_m\right|^2, \tag{3.19}$$

with

$$\Gamma(||B||,||A||_\infty)=4(||B||^2+||A||_\infty^2)^2+||A||_\infty^2(2||B||^2+4||A||_\infty^2). \tag{3.20}$$

Therefore

$$\|C\|_X^2 \leq 2\Gamma(||B||,||A||_\infty)\left(\sum_{n\in\mathbb{Z}}|B_n|^2\left|\sum_{m\in\mathbb{Z}}\rho^{|m-n|}\right|^2+\sum_{n\in S_A}\left|\sum_{m\in\mathbb{Z}}\rho^{|m-n|}B_m\right|^2\right) \tag{3.21}$$

$$\leq 2\Gamma(||B||,||A||_\infty)\left(\|B\|_X+\left\|\rho^{|\cdot|}*B\right\|_X\right) \tag{3.22}$$

$$\leq 2\Gamma(||B||,||A||_\infty)\left(\|B\|_X+\left\|\rho^{|\cdot|}\right\|_1\|B\|_X\right). \tag{3.23}$$

The last inequality follows from Young's inequality (3.12) with $u=\rho^{|\cdot|}$, $v=B$, $r=2$, $p=1$ and $q=2$. By equation (3.20) and inequality (3.23), $\|C\|_X$ tends to zero if $\|B\|_X$ tends to zero. ∎

Lemma 3.2. *Fix ω and $\kappa>0$. and consider the function $G:X\times\mathbb{R}\to X$ defined in (3.10). Then D_1G is continuous at $(\widetilde{A},0)$.*

Proof. It is enough to show that there exists β such that

$$\|[D_1G(B,\delta)-D_1G(0,0)]v\|\leq\beta\|v\|,\quad \forall v\in X, \tag{3.24}$$

and $\beta\to 0$ as $(B,\delta)\to 0\in X\times\mathbb{R}$. We have already determined $D_1G(0,0)$ in the proof of 3.1. We now calculate $D_1G(B,\delta)$ with $B\neq 0$, and $\delta\neq 0$, which (we check) is given by its partial derivatives. For $n\in\mathbb{Z}$ we have

$$\frac{\partial G_n}{\partial B_n}(B,\delta)=-2\delta+2\gamma\tanh\frac{\kappa}{2}\left(\sum_{m\in\mathbb{Z}}\rho^{-|m-n|}|B_m+\widetilde{A}_m|^2+2(B_n+\widetilde{A}_n)^2\right)+\omega, \tag{3.25}$$

$$\frac{\partial G_n}{\partial B_m}(B,\delta)=\delta+4\gamma\tanh\frac{\kappa}{2}\left(\rho^{|m-n|}|B_m+\widetilde{A}_m|\right)(B_n+\widetilde{A}_n),\quad m\in\{(n-1),(n+1)\}, \tag{3.26}$$

$$\frac{\partial G_n}{\partial B_m}(B,\delta)=4\gamma\tanh\frac{\kappa}{2}\left(\rho^{|m-n|}|B_m+\widetilde{A}_m|\right)(B_n+\widetilde{A}_n),\quad m\in\mathbb{Z}\setminus\{(n-1),n,(n+1)\}. \tag{3.27}$$

Let $v\in X$ and define $M=D_1G(B,\delta)-D_1G(0,0)$, $w=Mv$. Then for every $n\in\mathbb{Z}$ we have

$$w_n=\sum_{m\in\mathbb{Z}}M_{n,m}v_m=\sum_{m\in\mathbb{Z}}\left(\frac{\partial G_n}{\partial B_m}(B,\delta)-\frac{\partial G_n}{\partial B_m}(0,0)\right)v_m \tag{3.28}$$

$$=\delta(\Delta v)_n+2\gamma\tanh\frac{\kappa}{2}\sum_{m\in\mathbb{Z}}\rho^{|m-n|}\left(|B_m+\widetilde{A}_m|^2-|\widetilde{A}_m|^2\right)v_n \tag{3.29}$$

$$+4\gamma\tanh\frac{\kappa}{2}\sum_{m\in\mathbb{Z}}\rho^{|m-n|}\left(|B_m+\widetilde{A}_m|(B_n+\widetilde{A}_n)-|\widetilde{A}_m|\widetilde{A}_n\right)v_m \tag{3.30}$$

$$=\delta(\Delta v)_n+2|\gamma|\tanh\frac{\kappa}{2}(K_n+2J_n), \tag{3.31}$$

where

$$K_n=\sum_{m\in\mathbb{Z}}\rho^{|m-n|}\left(|B_m+\widetilde{A}_m|^2-\left|\widetilde{A}_m\right|^2\right)v_n,\quad \forall n\in\mathbb{Z}, \tag{3.32}$$

$$J_n = \sum_{m\in\mathbb{Z}} \rho^{|m-n|} \left(|B_m + \widetilde{A}_m|(B_n + \widetilde{A}_n) - |\widetilde{A}_m|\widetilde{A}_n \right) v_m, \quad \forall n \in \mathbb{Z}. \tag{3.33}$$

Hence

$$\|w\|_X \le |\delta|\,\|\Delta\|_{X\times X}\,\|v\|_X + 2|\gamma|\tanh\frac{\kappa}{2}\,\|K\|_X + 4|\gamma|\tanh\frac{\kappa}{2}\,\|J\|_X. \tag{3.34}$$

Observe that by equation (3.32)

$$|K_n| \le \sum_{m\in\mathbb{Z}} \rho^{|m-n|} \left(B_m^2 + 2|B_m||\widetilde{A}_m| \right) |v_n|,$$

so that

$$\begin{aligned}
||K||_X^2 &\le \sum_{n\in\mathbb{Z}} \left| \sum_{m\in\mathbb{Z}} \rho^{|m-n|} \left(B_m^2 + 2|B_m||\widetilde{A}_m| \right) |v_n| \right|^2 && (3.35)\\
&\le \sum_{n\in\mathbb{Z}} \|B\|_\infty (||B||_\infty + ||\widetilde{A}||_\infty) \left| \sum_{m\in\mathbb{Z}} \rho^{|m-n|} |v_n| \right|^2 && (3.36)\\
&\le \|B\|_\infty (||B||_\infty + ||\widetilde{A}||_\infty) \left\| \rho^{|\cdot|} \right\|_1^2 \|v\|_X^2, && (3.37)
\end{aligned}$$

using Young's inequality (3.12) with $u = \rho^{|\cdot|}$, $r=2$, $p=1$ and $q=2$. On the other hand, by equation (3.33) observe that

$$|J_n| \le \sum_{m\in\mathbb{Z}} \rho^{|m-n|} \left(|B_m||B_n| + |B_m||\widetilde{A}_n| + |B_n||\widetilde{A}_m| \right) |v_m|, \tag{3.38}$$

so that

$$\begin{aligned}
\|J\|_X^2 &\le \sum_{n\in\mathbb{Z}} \left| \sum_{m\in\mathbb{Z}} \rho^{|m-n|} \left(|B_m||B_n| + |B_m||\widetilde{A}_n| + |B_n||\widetilde{A}_m| \right) |v_m| \right|^2 && (3.39)\\
&\le \sum_{n\in\mathbb{Z}} \|B\|_\infty (||B||_\infty + 2||\widetilde{A}||_\infty) \left| \sum_{m\in\mathbb{Z}} \rho^{|m-n|} |v_m| \right|^2 && (3.40)\\
&\le \|B\|_\infty (||B||_\infty + 2||\widetilde{A}||_\infty) \left\| \rho^{|\cdot|} \right\|_1^2 \|v\|_X^2, && (3.41)
\end{aligned}$$

using again Young's inequality.

Using inequalities (3.35) and (3.40) in inequality (3.34) we have that

$$\|w\| \le \left(|\delta|\,\|\Delta\|_{X\times X} + 6|\gamma|\tanh\frac{\kappa}{2} ||B||(||B|| + 2||\widetilde{A}||_\infty)||\rho^{|\cdot|}||_1 \right) \|v\|. \tag{3.42}$$

By equation (3.29) and inequality (3.42) we thus have inequality (3.24) with β that vanishes as $(B,\delta) \to 0 \in X \times \mathbb{R}$, as required. ∎

4. Spectral analysis of 1-peak and shelf-type breathers

In this section we present results on the spectra of the linearization around some of the breather solutions considered in the previous sections and in [1]. We will consider the one-peak breather solutions, related to the minimizers of Section 2, and shelf-type breather of the local and nonlocal equation, studied in Section 3.

The linear stability analysis uses the fact that breather solutions are relative equilibria of equation (2.1) with respect to the action of global phase change. Using the variable v defined by $u=e^{-i\omega t}v$, Hamilton's equation (2.2) becomes

$$\dot{v}_n=-i\frac{\partial H_\omega}{\partial v_n^*}, \quad n\in Z, \quad \text{with} \quad H_\omega=H-\omega P. \tag{4.1}$$

A breather solution $u=e^{-i\omega t}A$ of equation (2.1) is a fixed point $v=A$ of equation (4.1).

Let $z=[q,p]^T\in X\times X$, with $z_n=[q_n,p_n]^T$, $q_n=\mathrm{Re}v_n$, $p_n=\mathrm{Im}v_n$, $n\in\mathbb{Z}$. Then equation (4.1) is also written as

$$\dot{z}=J\nabla h_\omega, \quad \text{with} \quad h_\omega=\frac{1}{2}H_\omega, \tag{4.2}$$

and $(Jz)_n=-[p_n,q_n]^T$, i.e. J is the standard symplectic operator in $X\times X$.

The linearization at a fixed point A of equation (4.1) is

$$\dot{z}=J\mathcal{H}z, \quad \text{with} \quad \mathcal{H}=\nabla^2 h_\omega(A), \tag{4.3}$$

i.e. $\mathcal{H}$ is the Hessian of h_ω at A.

To simplify the notation we set $\gamma=-1$ in this section. This does not affect the generality of the results.

We calculate $J\mathcal{H}$ explicitly as

$$J=\begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix}, \quad \mathcal{H}=\begin{bmatrix} L_+ & 0 \\ 0 & L_-\end{bmatrix}, \tag{4.4}$$

where

$$L_-=-\omega I-\delta\Delta+2\mathcal{A}, \quad L_+=-\omega I-\delta\Delta+2\mathcal{A}+4\mathcal{M}, \tag{4.5}$$

and $\mathcal{A}$, $\mathcal{M}$ are linear operators in X defined by

$$\mathcal{A}(n,k)=\tanh\frac{\kappa}{2}\Big(\sum_{m\in\mathbf{Z}} e^{-\kappa|m-n|}A_m^2\Big)\delta_{n,k}, \quad n,\kappa\in\mathbb{Z}, \tag{4.6}$$

$$\mathcal{M}(n,k)=\tanh\frac{\kappa}{2}e^{-\kappa|n-k|}A_kA_n, \quad n,k\in\mathbb{Z}, \tag{4.7}$$

with $\delta_{n,k}$ the Kronecker delta.

The linearization around breathers of the local cubic DNLS is obtained by taking the limit $\kappa\to\infty$ in equations (4.6) and (4.7).

The above calculations are formal. In the case $A\in l^\infty(\mathbb{Z},\mathbb{R})$, the operators $\mathcal{A}$, $\mathcal{M}$, L_-, L_+ are bounded operators in X, while J, $\mathcal{H}$, and $J\mathcal{H}$ are bounded operators in $X\times X$. We also see that L_-, L_+ and $\mathcal{H}$ are symmetric. Similar statements apply to the local case.

Let $X=l_2(\mathbb{Z},\mathbb{R})$ with the inner product $(u,v)=\sum_{n\in\mathbb{Z}}u_nv_n$. The norm is denoted by $||\cdot||$. Let X_c be the complexification of X, with the inner product $(u,v)_c=\sum_{n\in\mathbb{Z}}u_nv_n^*$.

Let $Y=l_2(\mathbb{Z},\mathbb{C})$ viewed as a real space with the (real) inner product $\langle u,v\rangle=\sum_{\mathbb{Z}}[(\mathrm{Re}u_n)(\mathrm{Re}v_n)+(\mathrm{Im}u_n)(\mathrm{Im}v_n)]$. Let $Y_c=l_2(\mathbb{Z},\mathbb{C}^2)$ with the inner product $\langle u,v\rangle_c=\sum_{\mathbb{Z}}[(\mathrm{Re}u_n)(\mathrm{Re}v_n)^*+(\mathrm{Im}u_n)(\mathrm{Im}v_n)^*]$. Y_c is the complexification of Y. Note that $Y=X\times X$, and $Y_c=X_c\times X_c$. The notation for X, Y is that of the previous sections, where we only used the Banach space structure.

Let M be a bounded operator in Y_c, or X_c. We denote the spectrum of M by $\sigma(M)$. The essential spectrum $\sigma_e(M)$ of M is defined as in [14], p.29, namely $\lambda \in \sigma(M)$ belongs to $\sigma_e(M)$ if $M-\lambda I$ either fails to be Fredholm, or is Fredholm with nonzero index. The definition of [15], p.243, requires a weaker condition, namely $M-\lambda I$ not semi-Fredholm.

Let $\mathbf{X}$ be a real or complex Hilbert space, with norm $||\cdot||_{\mathbf{X}}$, and let $\{\varepsilon_j\}_{j=1}^{\infty}$ be an orthonormal basis of $\mathbf{X}$. A bounded operator $T:\mathbf{X}\to\mathbf{X}$ is Hilbert–Schmidt in $\mathbf{X}$ if $\sum_{j=1}^{\infty}||T\varepsilon_j||_{\mathbf{X}}^2<\infty$ (this is independent of the choice of the basis). Recall that a Hilbert–Schmidt operator is compact.

4.1. Stability of one-peak breathers for small linear coupling. We will consider the one-peak breather solutions, defined below. These are solutions of equation (2.4) with $A \in X$, and start by showing some facts on arbitrary such solutions.

LEMMA 4.1. *Let $A\in X$. Then the operators $\mathcal{A}$, $\mathcal{M}$ defined in equations (4.6), (4.7) respectively are Hilbert–Schmidt in X, X_c.*

Proof. We use the basis $\{\hat{e}_n\}_{n\in\mathbb{Z}}$ of X and X_c. Both $\mathcal{A}$, $\mathcal{M}$ map $X\subset X_c$ to X, and it is enough to show the statement for X. First, by definition (4.6)

$$\sum_{n\in\mathbb{Z}}||\mathcal{A}\hat{e}_n||^2=\tanh^2\frac{\kappa}{2}\sum_{n\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}}e^{-\kappa|m-n|}A_m^2\right|^2=\tanh^2\frac{\kappa}{2}||g*A^2||_{l^2}^2, \tag{4.8}$$

where $g_n=e^{-\kappa|n|}$, and $(A^2)_n=A_n^2$, $\forall n\in\mathbb{Z}$. By the discrete Young's inequality (3.12) we have

$$||g*A^2||_{l^2}\le||g||_{l^1}||A^2||_{l^2}. \tag{4.9}$$

Since $||A||_{l^\infty}\le||A||$, both quantities are finite, and equation (4.8) and inequality (4.9) imply that $\mathcal{A}$ is Hilbert–Schmidt in X. Similarly, by definition (4.7) we have

$$\sum_{\lambda\in\mathbb{Z}}||\mathcal{M}\hat{e}_\lambda||^2=\tanh^2\frac{\kappa}{2}\sum_{\lambda\in\mathbb{Z}}A_\lambda^2\sum_{n\in\mathbb{Z}}|e^{-\kappa|\lambda-n|}A_n|^2\le||A||_{l^2}^4, \tag{4.10}$$

therefore $\mathcal{M}$ is also Hilbert–Schmidt in$X$. ∎

Lemma 4.1 is applied to fixed points $A\in X$ of equation (4.1). Define $\mathcal{H}_0$ as

$$\mathcal{H}_0=\begin{bmatrix}-\omega I-\delta\Delta & 0\\ 0 & -\omega I-\delta\Delta\end{bmatrix}. \tag{4.11}$$

Clearly $\mathcal{H}_0$ is a bounded operator in Y_c, moreover $J\mathcal{H}$ is a compact perturbation of $J\mathcal{H}_0$:

PROPOSITION 4.1. *Let $A\in X$, and let $J\mathcal{H}:Y\to Y$ be as in equations (4.4)-(4.7). Then (i) $\sigma_e(J\mathcal{H})=\sigma_e(J\mathcal{H}_0)$, and (ii) $\sigma_e(J\mathcal{H}_0)$ consists of $z\in\mathbf{C}$ with $Re z=0$, $Im z\in[-\omega,-\omega+4\delta]\cup[\omega-4\delta,\omega]$, for $\delta>0$, and $Re z=0$, $Im z\in[-\omega+4\delta,-\omega]\cup[\omega,\omega-4\delta]$, if $\delta<0$.*

Proof. **(Proof of Part (i).)** To show that $\sigma_e(J\mathcal{H})=\sigma_e(J\mathcal{H}_0)$ it is enough to show that $J\mathcal{H}-J\mathcal{H}_0$ is Hilbert–Schmidt in Y. We have

$$J\mathcal{H}-J\mathcal{H}_0=\begin{bmatrix}0 & 2\mathcal{A}\\ -2\mathcal{A}+4\mathcal{M} & 0\end{bmatrix}. \tag{4.12}$$

The Hilbert–Schmidt property in $Y = X \times X$ follows easily from Lemma 4.1. We can use the basis vectors $\hat{e}_n^j$, $n \in \mathbb{Z}$, $j = 1,2$, defined as $\hat{e}_n^1(k) = \delta_{n,k}[1,0]^T$, $\hat{e}_n^2(k) = \delta_{n,k}[0,1]^T$, for all n, $k \in \mathbb{Z}$. Part (ii) is shown in the next subsection. ◻

We now consider the stability of the 1−peak solution studied numerically in [1]. Consider a solution $A \in X$ of equation (2.4) with $\delta = 0$ that satisfies $A_n = 0$, for all $n \in \mathbb{Z} \setminus \{n_0\}$, and $A_{n_0} > 0$. The parameter $\kappa > 0$ is fixed and arbitrary. Such a solution is unique, and by Theorem 2.4 of [1] it can be continued uniquely to a C^0 family of solutions $A(\delta)$, $\delta \in (-\delta_0, \delta_0)$, of equation (2.4). We refer to these solutions as 1-peak breathers.

Remark 4.1. Note that the continuation result in [1] is for solutions with $||A(\delta)||_X$ fixed. Then ω is also a function of δ, and is not known a priori. The continuation result for fixed ω has a very similar proof, this is seen in the previous section.

For the $\delta = 0$ breather $A(0)$, $J\mathcal{H}$ is block diagonal with 2×2 blocks, each block corresponding to an integer $k \in \mathbb{Z}$. The block corresponding to $k = n_0$ has a double eigenvalue $\lambda_{n_0} = 0$. The other blocks have eigenvalues

$$\pm\lambda_k = \pm i\omega(1 - e^{\kappa|k-n_0|}), \quad \forall k \in \mathbb{Z} \setminus \{n_0\}, \tag{4.13}$$

with corresponding eigenfunctions $w_k^{\pm} \in Y_c$, given by

$$w_k^{\pm}(n) = \delta_{k,n}[1, \mp i]^T, \quad n \in \mathbb{Z}. \tag{4.14}$$

Thus $\sigma(J\mathcal{H})$ consists of a double zero eigenvalue, an infinite number of pairs of imaginary eigenvalues $\pm\lambda_k$, $\kappa > n_0$, of multiplicity 2, and their accumulation points $\pm i\omega$ (belonging to $\sigma_e(J\mathcal{H})$).

Considering now the linearized operator $J\mathcal{H}$ corresponding to a 1-peak breather $A(\delta)$, we have by Proposition 4.1 that $\sigma_e(J\mathcal{H}) = \sigma(J\mathcal{H}_0)$. The persistence of the double zero eigenvalue is a well-known fact, it follows from the fact that $L_- A(\delta) = 0$, the continuity of finite rank spectral projections, and the symmetry of spectra of symplectic operators. We now consider the persistence of the internal mode eigenvalues λ_k of equation (4.13). The linearized operator (4.4)-(4.7) corresponding to the 1-peak breather $A(\delta)$, $\delta \in (-\delta_0, \delta_0)$, will be denoted by $J\mathcal{H}(\delta)$.

Proposition 4.2. *Let $k > n_0$, and consider the pair of imaginary eigenvalues $\pm\lambda = \pm\lambda_k$ of the $J\mathcal{H}(0)$, as in equation (4.13). Then there exists a $\delta_{0,k} > 0$ for which the operators $J\mathcal{H}(\delta): Y \to Y$, $\delta \in (-\delta_{0,k}, \delta_{0,k})$, have two eigenvalues $\pm\lambda_j(\delta) \in i\mathbb{R}$, $j = 1,2$. The families $\pm\lambda_j(\delta)$, $j = 1,2$ are continuous in δ and satisfy $\pm\lambda_j(0) = \pm\lambda$.*

Proof. Fix $k > n_0$. By the continuity of finite dimensional spectral projections and the corresponding eigenvalues under perturbations by bounded operators, see [15], the families $\pm\lambda_j(\delta)$, $j = 1,2$ are continuous in δ, with $\pm\lambda(0) = \pm\lambda = \pm\lambda_k$, as stated, assuming that δ belongs to some interval $(-\delta_{0,k}, \delta_{0,k})$. Consider the eigenvalue $\lambda(0)$ of $J\mathcal{H}(0)$. By the symmetry of spectra of symplectic operators the two continued eigenvalues of $J\mathcal{H}(\delta)$ are either on the imaginary axis, or are off the imaginary axis and satisfy $\lambda_2(\delta) = -\lambda_1^*(\delta)$. To eliminate the second scenario we use the Krein dichotomy argument, see [14], p. 181, (7.14). In particular, if the $\lambda_j(\delta)$ have nonzero real parts then the corresponding eigenvectors $v_j(\delta)$ must satisfy

$$\langle \mathcal{H}(\delta)v_j(\delta), v_j(\delta)\rangle_c = 0, \quad \forall \delta \in (-\delta_{0,k}, \delta_{0,k}), \quad j = 1,2. \tag{4.15}$$

On the other hand, using explicit expressions for the eigenvectors $v_j(0)$, $j=1,2$, i.e. $v_1(0)=w_k^+$, $v_2(0)=w_m^+$, with $m<n_0$ satisfying $n_0-m=k-n_0$, and w_k^+, w_m^+ as in equation (4.14), we compute that

$$\langle \mathcal{H}(0)v_j(0), v_j(0)\rangle_c = -2\omega(1-e^{\kappa|k-n_0|}) \neq 0, \quad j=1,2, \tag{4.16}$$

contradicting equation (4.15) at $\delta=0$. ∎

We expect that for δ, $\gamma>0$, δ sufficiently small, the 1−peak breathers obtained by continuation should coincide with the global minimizers of Section 2. Nevertheless at the moment we can prove the following weaker statement.

Proposition 4.3. *Let $\lambda>0$, and fix $n_0\in\mathbb{Z}$. Consider the continuous one-parameter curve of $A(\delta)\in X$, $\delta\in[0,\delta_0]$, of solutions of equation (2.4) with $A(0)$ the 1-peak breather solution supported at $n=n_0$, and $||A(\delta)||=\lambda$, $\forall\delta\in[0,\delta_0]$. Then there is a sequence of $\{\delta_m\}_{m\in\mathbb{Z}^+}\subset[0,\delta_0]$, that converges to $\delta=0$, and such that $A(\delta_m)$ minimizes H of (2.3) over all configurations $u\in X$ with $P(u)=\lambda$.*

Proof. Fix $\lambda>0$. Let $H_\delta=H=\delta H_2-H_4$, where H_2, V are the quadratic, and quartic parts of H respectively, see equation (2.3). For each $\delta\in[0,\delta_0]$, let $u_*^\delta\in X$ be a minimizer of H_δ at power $P=\lambda$. For δ', $\delta\in[0,\delta_0]$, we have that $\delta<\delta'$ implies $H_\delta(u_*^\delta)\le H_\delta(u_*^{\delta'})\le H_{\delta'}(u_*^{\delta'})$. Also, we easily check that $\delta'\searrow\delta$ implies $H_{\delta'}(u_*^{\delta'})\to H_\delta(u_*^\delta)$. It follows that any sequence of points u_*^δ, $\delta\in[0,\delta_0]$, that converges to u_*^0 is a minimizing sequence for $H_0=V$, and therefore has a convergent subsequence, up to translations, by the arguments of Theorem 2.1. On the other hand, the critical points of V are solutions of equation (3.6). The support of minimizers with finite support consists of sets of consecutive sites by Theorem 2.2. Such solutions were described in [1], where we saw that all solutions with support consisting of more that three sites must have their maxima at interfaces. Then the only solution having the properties of Theorem 2.2 is the one supported on one site. We also observe that by Young's inequality, the right hand side of equation (3.6) maps X to X. Thus there are no solutions with infinite support. The points of the minimizing subsequence are solutions of equation (2.4), and must therefore come arbitrarily close to the breather $A(0)$, after suitable translations. By the uniqueness of the continuation, Theorem 2.4, [1], these points must eventually belong to the branch $A(\delta)$ of solutions obtained for some interval $[0,\delta_0]$. ∎

4.2. Essential spectrum of linearization around shelf-type breather: local case. We now consider the essential spectrum of the linearization around the shelf solutions of Section 3. We first consider the shelf solutions of the local (power) nonlinearity system. The main result is Theorem 4.1 below. It follows from an analysis of the piecewise linear discrete dynamical system defined by the linear operator. We give a fairly complete presentation of the theory. In the nonlocal case we have a weaker result, Theorem 4.2 below, using similar computations. The theory however is not complete.

In the case of the power nonlinearity system, we consider solutions obtained by continuation of the $\delta=0$ breather

$$\widetilde{A}_n = \begin{cases} \alpha, & \text{if } n\le 0,\\ 0, & \text{if } n\ge 1,\end{cases} \tag{4.17}$$

with $\alpha=\sqrt{\frac{\omega}{2}}$. This is a special case of the configuration of solution (3.3). The spectral results below are the same for the general case.

For ω fixed, let B_n, $n\in\mathbb{Z}$, be a solution of Theorem 3.1 and let

$$A_n=\widetilde{A}_n+B_n, \quad n\in\mathbb{Z}. \tag{4.18}$$

Let $\mathcal{L}_{\infty,\delta} = J\mathcal{H}$, with $J\mathcal{H}$ as in equation (4.4), L_-, L_+ as in equation (4.5), $\mathcal{A}$, $\mathcal{M}$ as in equations (4.6) and (4.7) with A_n as in equation (4.18), $\widetilde{A}_n$ as in equation (4.17), $\forall n \in \mathbb{Z}$. Operator $\mathcal{L}_{\infty,\delta}$ is therefore the formal linearization around the shelf solution of Theorem 3.1.

Also let $\mathcal{L} = J\mathcal{H}$, with $J\mathcal{H}$ as in equation (4.4), with L_-, L_+ as in equation (4.5), and

$$\mathcal{A}(n,k) = \mathcal{A}_0(n,k) = \widetilde{A}_n^2 \delta_{n,k}, \quad \mathcal{M}(n,k) = \mathcal{M}_0(n,k) = \widetilde{A}_n^2 \delta_{n,k}, \tag{4.19}$$

where $\widetilde{A}_n$ is as in equation (4.17).

Proposition 4.4. *The operator $\mathcal{L}_{\infty,\delta}$ is a compact perturbation of $\mathcal{L}$, and therefore $\sigma_e(\mathcal{L}_{\infty,\delta}) = \sigma_e(\mathcal{L})$ in Y.*

The prove the statement it is enough to show that $\mathcal{A} - \mathcal{A}_0$, $\mathcal{M} - \mathcal{M}_0$ are Hilbert–Schmidt. The proof of this fact is a simplified version of the proof of Lemma 4.5 for the nonlocal problem below, and is omitted.

To calculate the essential spectrum of $\mathcal{L}$ we study the equation $(\mathcal{L} - \lambda)z = 0$, $z = [q,p]^T$, with $z_n = [q_n, p_n]^T$ as a discrete dynamical system. The (spectral) parameter λ varies over $\mathbb{C}$. By the definition of $\mathcal{L}$, with $\mathcal{A}$, $\mathcal{M}$ of equation (4.19), and $\widetilde{A}$ of equation (4.17) we have that $\mathcal{L}z = \lambda z$ is equivalent to

$$-\delta(p_{n+1} + p_{n-1} - 2p_n) = \lambda q_n, \quad -2\omega q_n + \delta(q_{n+1} + q_{n-1} - 2q_n) = \lambda p_n, \quad n \leq 0; \tag{4.20}$$

$$-\omega p_n - \delta(p_{n+1} + p_{n-1} - 2p_n) = \lambda q_n, \quad \omega q_n + \delta(q_{n+1} + q_{n-1} - 2q_n) = \lambda p_n, \quad n \geq 1. \tag{4.21}$$

Finite difference equation (4.20), (4.21) is furthermore equivalent to a first order system $y_{n+1} = \mathcal{F}(y_n)$ for a sequence of vectors $y_n = [y_n(1), \ldots, y_n(4)]^T \in \mathbb{C}^4$, $n \in \mathbb{Z}$. This formulation uses the identification

$$[y_n(1), y_n(2), y_n(3), y_n(4)]^T = [q_{n-1}, q_n, p_{n-1}, p_n]^T. \tag{4.22}$$

By equation (4.22), $n \leq 0$, equation (4.20) is then equivalent to

$$y_{n+1} = M_-(\lambda) y_n, \quad n \leq 0, \quad \text{with} \quad M_-(\lambda) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 2\frac{\omega}{\delta} + 2 & 0 & \frac{\lambda}{\delta} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{\lambda}{\delta} & -1 & 2 \end{bmatrix}. \tag{4.23}$$

Similarly by equation (4.22), $n \geq 1$, equation (4.21) is equivalent to

$$y_{n+1} = M_+(\lambda) y_n, \quad n \geq 1, \quad \text{with} \quad M_+(\lambda) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -\frac{\omega}{\delta} + 2 & 0 & \frac{\lambda}{\delta} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{\lambda}{\delta} & -1 & -\frac{\omega}{\delta} + 2 \end{bmatrix}. \tag{4.24}$$

Equations (4.23) and (4.24) define a piecewise linear discrete-time dynamical system that may be written as $y_{n+1} = M(\lambda) y_n$, where $M(\lambda) = M_-(\lambda)$ if $n \leq 0$, and $M(\lambda) = M_+(\lambda)$ if $n \geq 1$.

Proposition 4.5. *Given any $m \in \mathbb{Z}$, $a \in \mathbb{C}^4$, there exists a unique y_n, $n \in \mathbb{Z}$, satisfying equations (4.23) and (4.24) and $y_m = a$.*

Proof. We check first that the $M_\pm(\lambda)$ are invertible, $\forall\lambda\in\mathbb{C}$. By equations (4.23) and (4.24)

$$P_-(r;\lambda)=\det(M_-(\lambda)-rI)=[(r-1)^2-2r\frac{\omega}{\delta}](r-1)^2+r^2\frac{\lambda^2}{\delta^2}, \tag{4.25}$$

$$P_+(r;\lambda)=\det(M_+(\lambda)-rI)=[-r(2-\frac{\omega}{\delta}-r)+1]^2+r^2\frac{\lambda^2}{\delta^2}. \tag{4.26}$$

We see that $P_\pm(r,\lambda)=0$, $r=0$ leads to a contradiction, $\forall\lambda\in\mathbb{C}$. Thus the $M_\pm(\lambda)$ are invertible, $\forall\lambda\in\mathbb{C}$. This fact allows us to iterate conditions (4.23) and (4.24) backwards as well, writing

$$[M_-(\lambda)]^{-1}y_{n+1}=y_n, \quad n\le 0, \tag{4.27}$$

$$[M_+(\lambda)]^{-1}y_{n+1}=y_n, \quad n\ge 1. \tag{4.28}$$

We can thus iterate both forwards and backwards in each segment. We check that equations (4.23), (4.24), (4.27), and (4.28) allow us to cross the interface separating the $M_\pm(\lambda)$ segments in a unique way, in either direction. ∎

We examine the dynamics of each of the systems (4.23) and (4.24), particularly its dependence on the parameter λ. In both cases we can start with some y_1 and consider the iteration backwards, and forwards respectively. Since each system is linear this amounts to a spectral analysis of the matrices $M_\pm(\lambda)$. The results of this analysis are then interpreted in operator-analytic terms for the resolvent equation $(\mathcal{L}-\lambda)z=f$, $f=[g,h]^T\in X\times X\subset Y_c$.

Consider a linear system $x_{n+1}=Tx_n$ in $\mathbb{C}^N$, where T an invertible $N\times N$ matrix, and n belongs to a set of indices $\mathcal{J}$ that is either $\mathbb{Z}$, or the set of all integers above or below some $m\in\mathbb{Z}$. The system is *hyperbolic* if there exist subspaces E^s, E^u of $\mathbb{C}^N$, $\dim(E^s)+\dim(E^u)=N$ that are invariant under T, and for which there exist $C>0$, $\varrho>1$ such that

$$||T^nP^sv||_{\mathbb{C}^N}\le C\varrho^{-n}||P^sv||_{\mathbb{C}^N}, \quad ||T^{-n}P^uv||_{\mathbb{C}^N}\le C\varrho^{-n}||P^uv||_{\mathbb{C}^N}, \quad \forall n\in\mathcal{J}, \tag{4.29}$$

where P^s, P^u are respective projections to E^s, E^u (and commute with T). The subspaces E^s, E^u are referred to as *stable* and *unstable* subspaces respectively. Hyperbolicity can be often verified by calculating the eigenvalues of T. The dimensions of E^s, E^u are determined be the multiplicities of eigenvalues inside and outside the unit circle respectively.

In the present problem hyperbolicity will be examined for each of the systems (4.23) and (4.24). The corresponding stable and unstable subspaces will be denoted by $E^s_\pm$, $E^u_\pm$. Note that the E^s_-, E^u_- for system (4.23) refer to the forward iteration of $M_-(\lambda)$.

Lemmas 4.2, 4.3 connect the analysis of the eigenvalues of the $M_\pm(\lambda)$ to the of spectrum of $\mathcal{L}$. The proofs are at the end of the subsection.

Lemma 4.2. *Let $\lambda\in\mathbb{C}$ be such that the systems (4.23) and (4.24) defined by $M_-(\lambda)$, $M_+(\lambda)$ respectively are both hyperbolic, with $\dim(E^u_-)+\dim(E^s_+)=N$, $E^u_-\cap E^s_+=\{0\}$. Then λ belongs to the residual set of $\mathcal{L}$ in Y.*

Lemma 4.3. *Let $\lambda\in\mathbb{C}$, and suppose that one of $M_-(\lambda)$, $M_+(\lambda)$ has a semisimple eigenvalue on the unit circle. Then λ belongs to the essential spectrum of $\mathcal{L}$ in Y.*

Define the sets

$$B_1 = \left\{ -(\omega - 4\delta \sin^2 \frac{k}{2})^2 : \quad k \in \mathbb{R}) \right\}, \tag{4.30}$$

$$B_2 = \left\{ -4\delta \sin^2 \frac{k}{2}(2\omega + 4\delta \sin^2 \frac{k}{2}) : \quad k \in \mathbb{R}) \right\}. \tag{4.31}$$

We will further assume that B_1, B_2 are disjoint. By definitions (4.30) and (4.31) this property is clearly satisfied for $|\delta|$ sufficiently small.

LEMMA 4.4. *Let $\lambda \in \mathbb{C}$. If $\lambda^2 \notin B_1 \cup B_2$ then λ either belongs to the residual set of $\mathcal{L}: Y \to Y$, or is an eigenvalue of $\mathcal{L}$ of finite multiplicity.*

THEOREM 4.1. *The essential spectrum of the operators $\mathcal{L}$ and $\mathcal{L}_{\infty,\kappa}$ in Y consists of the set of $\lambda \in \mathbb{C}$ satisfying $\lambda^2 \in B_1 \cup B_2$.*

The case $\lambda^2 \in B_1$ yields two intervals of λ on the imaginary axis, and coincides with the essential spectrum of the linearization around breathers that decay at infinity, as in Proposition 4.1 (its proof is completed below).

The case $\lambda^2 \in B_2$ yields two intervals that are both on either the imaginary or the real axis, depending on δ and ω. For instance, for $|\delta| < \frac{|\omega|}{2}$ we have stability for $\delta\omega > 0$, and instability for $\delta\omega < 0$. In both cases zero is one the endpoints of the intervals, and is included in $\sigma_e(\mathcal{L})$.

Proof. **(Proof of Lemma 4.4.)** The eigenvalues of $M_\pm(\lambda)$ are the roots of the polynomials $\det(M_\pm(\lambda) - rI)) = P_\pm(r;\lambda)$, shown in equations (4.25) and (4.26).

We can rewrite $P_-(r;\lambda) = 0$ as

$$\lambda^2 = \delta(r + r^{-1} - 2)[2\omega - \delta(r + r^{-1} - 2)]. \tag{4.32}$$

If a root r is on the unit circle, i.e. $r = e^{ik}$ for some real k, then λ^2 must belong to B_1. Moreover the roots of P_- come in pairs r, r^{-1}. If $\lambda^2 \notin B_1$ we therefore have two roots inside and two roots outside the unit circle. The system defined by $M_-(\lambda)$ is hyperbolic, and $\dim(E^s_-) = \dim(E^u_-) = 2$.

Analogously, $P_+(r;\lambda) = 0$ can be written as

$$\lambda^2 = -[\delta^2(r + r^{-1})^2 + (\omega - 2\delta)^2 + 2\delta(r + r^{-1})]. \tag{4.33}$$

It follows that if $r = e^{ik}$ for some real k, then λ^2 must belong to B_2. Similarly, since the roots of P_+ come in pairs r, r^{-1}, if $\lambda^2 \notin B_2$ then the system defined by $M_+(\lambda)$ is hyperbolic, and $\dim(E^s_+) = \dim(E^u_+) = 2$.

Therefore $\lambda^2 \notin B_1 \cup B_2$ implies $\dim(E^u_-) + \dim(E^s_+) = 4$. If in addition $E^u_- \cap E^s_+ = \{0\}$, Lemma 4.2 implies that λ is in the residual set of $\mathcal{L}$. If otherwise $E^u_- = E^s_+$ then $(\mathcal{L} - \lambda)z = 0$ has exponentially decaying solutions and is therefore an eigenvalue of $\mathcal{L}$. The set of these solutions is clearly two-dimensional. ∎

Proof. **(Proof of Theorem 4.1.)** By the calculation in the proof of Lemma 4.4, if λ^2 belongs to $B_1 \cup B_2$ then either $M_-(\lambda)$ or $M_+(\lambda)$ has an eigenvalue of the form $r = e^{ik}$ for some real k. By Lemma 4.3 then λ belongs to the essential spectrum of $\mathcal{L}$. If, on the other hand, $\lambda^2 \notin B_1 \cup B_2$, Lemma 4.4 implies that λ can not belong to $\sigma_e(\mathcal{L})$. ∎

Proof. **(Proof of Proposition 4.1, Part (ii).)** Solving $J\mathcal{H}_0 u = \lambda u$ explicitly i.e. using $u_n = e^{ikn}$, $\sigma_e(J\mathcal{H}_0)$ is as in the statement by Lemma 4.3. The result coincides with the part of the essential spectrum of $\mathcal{L}$ that comes from the set B_1 of definition (4.30). ☐

We now prove Lemmas 4.2, and 4.3.

Proof. **(Proof of Lemma 4.2.)** The resolvent equation $(\mathcal{L}-\lambda)z=f$, $z=[p,q]^T$, $f=[g,h]^T$ in $X\times X\subset Y_c$ is equivalent to

$$L_-p=\lambda q+g, \quad -L_+q=\lambda p+h, \tag{4.34}$$

and can be also written in the dynamical system form

$$y_{n+1}=My_n+F_n, \quad \text{with} \quad F_n=[0,h_n,0,-g_n]^T, \tag{4.35}$$

where $M=M_-(\lambda)$ if $n\le 0$, $M=M_+(\lambda)$ if $n\ge 1$. The solution of equation (4.35) is obtained by iterating forwards or backwards in time. We will examine forward iterates, from $n=1$ to $n>1$ arbitrarily large, similar considerations apply to backward iterates from $n=1$.

By equation (4.35) the solution for $n\ge 1$ is

$$y_{n+1}=M^n y_1+\sum_{j=1}^{n-1}M^jF_{n-j}, \quad \text{with} \quad M=M_+(\lambda). \tag{4.36}$$

Using the assumption of hyperbolicity, we can use projections P^s, P^u to the stable and unstable subspaces respectively to write equation (4.36) as

$$y_{n+1}=M^nP^sy_1+\sum_{j=1}^{n-1}M^jP^sF_{n-j}+M^nP^uy_1+\sum_{j=1}^{n-1}M^jP^uF_{n-j}, \quad \forall n>1. \tag{4.37}$$

We first claim that if $[g,h]^T$ belongs to $X_c\times X_c$ then the first two terms in the right hand side of equation (4.37) belong to $X^4_{c,+}=X_{c,+}\times X_{c,+}\times X_{c,+}\times X_{c,+}$, where $X_{c,+}$ is the space of configurations in X_c that vanish for $n\le 0$.

The claim is immediate for the first term, since $||M^nP^sy_1||_{\mathbb{C}^4}$ decays exponentialy in n by the hyperbolicity assumption, see inequality (4.29). Denote the second term by G_n, then by equation (4.37)

$$G_n=\sum_{j=1}^{n-1}M^jP^sF_{n-j}=\sum_{k=1}^{n}M^{n-k}P^sF_k, \tag{4.38}$$

therefore by the hyperbolicity assumption, see inequality (4.29),

$$||G_n||_{\mathbb{C}^4}\le C\sum_{k=1}^{n}\rho_2^{-|n-k|}||F_k||_{\mathbb{C}^4}\le C\sum_{k=1}^{\infty}\rho_2^{|n-k|}||F_k||_{\mathbb{C}^4}, \tag{4.39}$$

for some $C>0$, and $\rho_2>1$. Note that $[g,h]^T\in X_c\times X_c$ implies $F\in X_c^4$. Let $\tilde{f}_k=||F_k||_{\mathbb{C}^4}$ for $k\ge 1$, $\tilde{f}_k=0$ for $k\le 0$, and define

$$\tilde{g}_n=C\sum_{k\in\mathbb{Z}}\rho_2^{-|n-k|}\tilde{f}_k, \quad n\in\mathbb{Z}. \tag{4.40}$$

Then $\tilde{g}_n \leq ||G_n||_{\mathbb{C}^4}$ for all $n \geq 1$, and by inequality (4.39), and Young's inequality (3.12),

$$\sum_{n=1}^{\infty} ||G_n||^2_{\mathbb{C}^4} \leq ||\tilde{g}||^2_{l^2} \leq C^2 ||\rho_2^{-|\cdot|}||^2_{l^1} ||\tilde{f}||^2_{l^2} = C^2 ||\rho_2^{-|\cdot|}||^2_{l^1} ||F||^2_{X_c^4}. \tag{4.41}$$

Thus the second term of the right hand side of equation (4.37) belongs to $X^4_{c,+}$, as claimed.

Consider the third and fourth terms in the right hand side of equation (4.37). They can be written as $M^n B_n$ with

$$B_n = P^u y_1 + \sum_{j=1}^{n-1} M^{-(n-j)} P^u F_{n-j}, \tag{4.42}$$

where $M^{-p} = (M^{-1})^p$ if p a positive integer. To have the sequence of y_n of equation (4.37) in $X^4_{c,+}$ we need that $B_n \to 0$ as $n \to \infty$. By equation (4.42) we therefore need $P^u y_1 = b_+$, with

$$b_+ = -\lim_{n\to\infty} \sum_{j=1}^{n-1} M^{-(n-j)} P^u F_{n-j}. \tag{4.43}$$

We will check below that the limit exists, and that the solution of equation (4.36), $n \geq 1$, obtained for such y_1 belongs to $X^4_{c,+}$.

To show that the limit in equation (4.43) exists we must check that

$$I_n = \sum_{j=0}^{n-1} M^{-(n-j)} P^u F_{n-j} = \sum_{k=1}^{n} M^{-k} F_k \tag{4.44}$$

converges. For $n > m$ we have

$$||I_n - I_m||_{\mathbb{C}^4} \leq \sum_{k=m+1}^{n} ||M^{-k} P^u F_k||_{\mathbb{C}^4} \leq C \sum_{k=m+1}^{n} \rho_2^k ||F_k||_{\mathbb{C}^4} \tag{4.45}$$

for some $\rho_2 < 1$, $C > 0$, by the hyperbolicity assumption. Then

$$||I_n - I_m||_{\mathbb{C}^4} \leq C\Big(\sum_{k=m+1}^{n} \rho_2^{2k}\Big)^{\frac{1}{2}} ||F||_{X^4} \leq C \rho_2^{m-1} C_{\rho_2} ||F||_{X_c^4}, \tag{4.46}$$

where C_{ρ_2} is independent of n, m. Thus I_n converges as $n \to \infty$, and b_+ in equation (4.43) is well defined.

To check that the sum J_n of the third and fourth terms in the right hand side of equation (4.37) obtained by choosing y_1 as above belongs to X^4_+, we note that for $n \geq 1$,

$$J_n = M^n \Big[\lim_{k\to\infty} \sum_{j=0}^{k-1} M^{-(k-j)} P^u F_{k-j} - \sum_{j=0}^{n-1} M^{-(n-j)} P^u F_{k-j}\Big] \tag{4.47}$$

$$= M^n \Big[\sum_{l=1}^{+\infty} M^{-l} P^u F_l - \sum_{l=1}^{n} M^{-l} P^u F_l\Big] \tag{4.48}$$

$$= \sum_{l=n+1}^{+\infty} M^{-(l-n)} P^u F_l \tag{4.49}$$

$$= \sum_{k=1}^{+\infty} M^{-k} P^u F_{k+n}. \tag{4.50}$$

Then

$$||J_n||_{\mathbb{C}^4} \leq \sum_{k=1}^{+\infty} ||M^{-k} P^u F_{k+n}||_{\mathbb{C}^4} \leq C \sum_{k=1}^{+\infty} \rho_3^{-k} ||F_{k+n}||_{\mathbb{C}^4}, \tag{4.51}$$

with $\rho_3 > 1$, $C > 0$, using the hyperbolicity assumption. Therefore

$$||J_n||_{\mathbb{C}^4} \leq C \sum_{-\infty}^{l=-1} \rho_3^{l} ||F_{n-l}||_{\mathbb{C}^4}, \quad \forall n \geq 1. \tag{4.52}$$

Letting $\alpha_l = \rho_3^{-|l|}$ if $l \leq -1$, $\alpha_l = 0$ if $l \geq 0$, and $\beta_m = ||F_m||_{\mathbb{C}^4}$ if $m \geq 1$, $\beta_m = 0$ if $m \leq 0$, we then have that $\tilde{\zeta}_n$, given by

$$\tilde{\zeta}_n = C \sum_{l \in \mathbb{Z}} \alpha_l \beta_{n-l} \tag{4.53}$$

is well defined for all $n \in \mathbb{Z}$, and satisfies $\tilde{\zeta}_n = ||J_n||_{\mathbb{C}^4}$. We then have

$$\sum_{n=1}^{+\infty} ||J_n||^2_{\mathbb{C}^4} \leq ||\tilde{\zeta}||^2_{l^2} \leq C ||\rho_3^{-|\cdot|}||^2_{l^1} ||F||^2_{X_c^4} \tag{4.54}$$

by Young's inequality (3.12). The left hand side is then finite, as required.

By equation (4.43), the existence of the limit implies $b_+ \in E_+^u$. Then $P^u y_1 = b_+$ implies that $y_1 = b_+ + v_+^s$, where v_+^s is an arbitrary element of E_+^s.

Similar arguments apply for the backwards solution of equation (4.36), starting from y_1, and iterating backwards for $n \leq 1$. We show that the projection to E_-^u leads to a part of the solution that belongs to $X_{c,-}^4$ (where $X_{c,-}$ consists of sequences in X_c that vanish for $n > 0$), while projection to E_-^s leads to a term that can only decay if y_1 satisfies $y_1 = b_- + v_-^u$, with $b_- \in E_-^s$ unique, and v_-^u an arbitrary element of E_-^u.

Thus the two restrictions on y_1 imply that it belongs to two affine subspaces $b_+ + E_+^s$, $b_- + E_-^u$. By the hypothesis on E_+^s, E_-^u, their intersection exists and is unique.

It is also easy to check that the solution $[p,q]^T$ of equation (4.34) obtained by gluing the forward and backward solution of equation (4.35) belongs to $X_c \times X_c$ if and only if the forward and backward solution of equation (4.35) belong to $X_{c,+}^4$, $X_{c,-}^4$ respectively. □

Proof. **(Proof of Lemma 4.3.)** We will assume that $M_+(\lambda)$ has a semisimple eigenvalue r_c on the unit circle, the other case is treated similarly. Fix an integer $m > 4$, and $N = 2m+1$. Let $\tilde{z}_n^N$, $n \in \mathbb{Z}$ satisfy $\tilde{z}_n^N = 0$, for all $n \leq 0$, $n > N$,and $\tilde{z}_{n+1} = M_+ \tilde{z}_n$, for all $1 \leq n \leq N$, with $\tilde{z}_1^N \in E_+(r_c)$, the corresponding invariant subspace of $y_{n+1} = M_+ y_n$.

By equations (4.20) and (4.21) we have that $[(L-\lambda)\tilde{z}^N]_n = 0$ for all n, except $n = 0$, 1, N, and $N+1$. Also $y_1 \in E_+(r_c)$ implies that $||y_{n+1}||_{\mathbb{C}^4} = ||y_n||_{\mathbb{C}^4}$, $\forall n \geq 1$. Then

$$|q_2|^2 + |p_2|^2 = |q_{2p}|^2 + |p_{2p}|^2, \quad |q_1|^2 + |p_1|^2 = |q_{2p+1}|^2 + |p_{2p+1}|^2, \quad \forall p \geq 1. \tag{4.55}$$

Letting $|q_1|^2+|p_1|^2=R_1^2$, $|q_2|^2+|p_2|^2=R_2^2$, we calculate

$$||\tilde{z}^N||^2_{Y_c}=(m+1)R_1^2+mR_2^2, \tag{4.56}$$

and

$$|[(L-\lambda)\tilde{z}^N]_N|\leq 3(|\omega+2|^2+|\lambda|^2)R_1^2+3\delta^2R_2^2, \quad [(L-\lambda)\tilde{z}^N]_{N+1}=\delta^2R_1^2, \tag{4.57}$$

furthermore

$$[(L-\lambda)\tilde{z}^N]_0=[(L-\lambda)\tilde{z}^N]_{N+1}, \quad [(L-\lambda)\tilde{z}^N]_1=[(L-\lambda)\tilde{z}^N]_N. \tag{4.58}$$

Letting $\tilde{u}^N=||\tilde{z}^N||^{-1}_{Y_c}\tilde{z}^N$ we therefore have

$$||(L-\lambda)\tilde{u}^N||_{Y_c}\leq\frac{C}{N}, \tag{4.59}$$

with C depending on R_1, R_2, δ, ω, and λ. We therefore have a sequence of $\tilde{u}^N$, $N>9$, that belongs to $\mathrm{Ker}^{\perp}(L-\lambda)$ and satisfies

$$||\tilde{u}^N||_{Y_c}=1, \quad \forall N>0, \quad \text{and} \quad ||(L-\lambda)\tilde{u}^N||_{Y_c}\to 0 \quad \text{as} \quad N\to\infty. \tag{4.60}$$

Then $L-\lambda$, restricted to $\mathrm{Ker}^{\perp}(L-\lambda)$, can not have a closed range, this would imply the existence of a bounded inverse, contradicting property (4.60). Thus $L-\lambda$ can not be Fredholm. □

4.3. Essential spectrum of linearization around shelf-type breather: nonlocal case. We now examine the linearization around the solutions of Theorem 3.2 for the nonlocal case. We only analyze the essential spectrum. As in the local case we will compute the essential spectrum of a simpler operator that is a compact perturbation of the operator we are interested in. This simpler operator is also nonlocal, and we do not have a ready analogy with dynamical systems. This difference also yields a weaker result for the essential spectra of nonlocal cases, see Theorem 4.2.

To simplify the notation we will consider solutions obtained by continuation of the $\delta=0$ (finite κ) breather

$$\widetilde{A}_n=\begin{cases}\alpha, & \text{if } n<0\\ \frac{\alpha}{\sqrt{1-\rho}}, & \text{if } n=0\\ 0, & \text{if } n>0,\end{cases} \tag{4.61}$$

with $\alpha=\sqrt{-\frac{\omega}{2\gamma}}$, $\rho=e^{-\kappa}$. This is a special case of the configuration of solution (3.9). For ω, κ fixed, let B_n, $n\in\mathbb{Z}$, be a solution of Theorem 3.2 and let

$$A_n=\widetilde{A}_n+B_n, \quad n\in\mathbb{Z}, \tag{4.62}$$

with $\widetilde{A}$ as in equation (4.61).

Let $\mathcal{L}_{\kappa,\delta}=J\mathcal{H}$, with $J\mathcal{H}$ as in equation (4.4), L_-, L_+ as in equation (4.5), $\mathcal{A}$, $\mathcal{M}$ as in equations (4.6) and (4.7), with A_n as in definition (4.62). Operator $\mathcal{L}_{\kappa,\delta}=J\mathcal{H}$ is the linearization around the shelf-type solution of Theorem 3.2.

To calculate the essential spectrum of $\mathcal{L}_{\kappa,\delta}$ we will use its perturbation $\overline{\mathcal{L}}_2$, defined by $\overline{\mathcal{L}}_2=J\mathcal{H}$, where $J\mathcal{H}$ is as in equation (4.4) with L_-, L_+ as in equation (4.5), and $\mathcal{A}$, $\mathcal{M}$ given by

$$\mathcal{A}(n,k)=\overline{\mathcal{A}}_2(n,k)=\tanh\frac{\kappa}{2}A_n^2\delta_{n,k}, \quad \mathcal{M}(n,k)=\overline{\mathcal{M}}_2(n,k)=\tanh\frac{\kappa}{2}A_nA_me^{-\kappa|m-n|}\delta_{n,k}, \tag{4.63}$$

for all k, $n\in\mathbb{Z}$, where A_n, $n\in\mathbb{Z}$, is as in equation (4.17), i.e. A_n is the shelf solution of the local problem with $\delta=0$. We also define the operators $\overline{\mathcal{A}}_1$, $\overline{\mathcal{M}}_1$ by

$$\overline{\mathcal{A}}_1(n,k)=\tanh\frac{\kappa}{2}\widetilde{A}_n^2\delta_{n,k},\quad \overline{\mathcal{M}}_1(n,k)=\tanh\frac{\kappa}{2}\widetilde{A}_n\widetilde{A}_m e^{-\kappa|m-n|}\delta_{n,k}, \tag{4.64}$$

for all k, $n\in\mathbb{Z}$, where $\widetilde{A}_n$ is as in equation (4.61). We easily see that $\overline{\mathcal{A}}_1-\overline{\mathcal{A}}_2$, $\overline{\mathcal{M}}_1-\overline{\mathcal{M}}_2$ are compact. For instance,

$$(\overline{\mathcal{M}}_1-\overline{\mathcal{M}}_2)(0,n)=(\overline{\mathcal{M}}_1-\overline{\mathcal{M}})(n,0)=\tanh\frac{\kappa}{2}\alpha^2((1-\rho)^{-1}-1)e^{-\kappa|n-1|},\quad \forall n\leq 0, \tag{4.65}$$

while all other entries of $\overline{\mathcal{M}}_1-\overline{\mathcal{M}}_2$ vanish. It easily follows that $\overline{\mathcal{M}}_1-\overline{\mathcal{M}}_2$ is Hilbert–Schmidt.

Lemma 4.5. *The operator $\mathcal{L}_{\kappa,\delta}:Y\to Y$ is a compact perturbation of the operator $\overline{\mathcal{L}}_2:Y\to Y$.*

Proof. To show that $\mathcal{L}_{\kappa,\delta}-\tilde{\mathcal{L}}$ is compact it is enough to show that $\mathcal{A}-\overline{\mathcal{A}}_2$, and $\mathcal{M}-\overline{\mathcal{M}}_2$ are compact.

The (squared) Hilbert–Schmidt norm of $T=\mathcal{A}-\overline{\mathcal{A}}_2$ is

$$\sum_{n\in\mathbb{Z}}|T\hat{e}_n|^2=\sum_{n\in\mathbb{Z}}\left|\tanh\frac{\kappa}{2}\sum_{m\in\mathbb{Z}}e^{\kappa|m-n|}|\widetilde{A}_m+B_m|^2-\overline{A}_n^2\right|^2, \tag{4.66}$$

where $\overline{A}_n=\frac{\alpha}{2}$, if $n\leq 0$, $\overline{A}_n=0$, if $n>0$. Using the equation (3.6) satisfied by $\widetilde{A}$, equation (4.66) becomes

$$\begin{aligned}\sum_{n\in\mathbb{Z}}|T\hat{e}_n|^2&=\sum_{n\in\mathbb{Z}}2\left|\tanh^2\frac{\kappa}{2}2\sum_{m\in\mathbb{Z}}e^{\kappa|m-n|}|\widetilde{A}_mB_m|+\sum_{m\in\mathbb{Z}}e^{\kappa|m-n|}|B_m|^2\right|^2 &(4.67)\\ &\leq 2|\tanh^2\frac{\kappa}{2}\left[2\left(\frac{\alpha}{1-\rho}\right)^2||e^{-\kappa|\cdot|}*|B|||_{l^2}^2+||e^{-\kappa|\cdot|}*|B|^2||_{l^2}^2\right], &(4.68)\end{aligned}$$

where $|B|_n=|B_n|$, $(|B|^2)_n=|B_n|^2$, $\forall n\in\mathbb{Z}$. We have also used $|\widetilde{A}_n|\leq\frac{\alpha}{1-\rho}$, $\forall n\in\mathbb{Z}$. The two terms of equation (4.68) are finite by Young's inequality, and the fact that $B\in l^2$. Thus T is Hilbert–Schmidt. Since $\overline{\mathcal{M}}_1-\overline{\mathcal{M}}_2$ is a Hilbert–Schmidt, it is sufficient to show that $\mathcal{M}-\overline{\mathcal{M}}_1$ is compact. The (squared) Hilbert–Schmidt norm of $S=\mathcal{M}-\overline{\mathcal{M}}_1$ is

$$\begin{aligned}\sum_{n\in\mathbb{Z}}|S\hat{e}_n|^2&=\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|\tanh\frac{\kappa}{2}e^{-\kappa|m-n|}(\widetilde{A}_m+B_m)(\widetilde{A}_n+B_n)-\tanh\frac{\kappa}{2}\widetilde{A}_m\widetilde{A}_n|^2 &(4.69)\\ &\leq 3\tanh^2\frac{\kappa}{2}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left(2|e^{-\kappa|m-n|}\widetilde{A}_nB_m|^2+|e^{-\kappa|m-n|}B_nB_m|^2\right) &(4.70)\\ &\leq 3|\tanh^2\frac{\kappa}{2}\left[2\left(\frac{\alpha}{1-\rho}\right)^2||e^{-\kappa|\cdot|}*|B|||_{l^2}^2+||e^{-\kappa|\cdot|}||_{l^2}^2||B||_{l^2}^2\right], &(4.71)\end{aligned}$$

using also $|\widetilde{A}_n|\leq\frac{\alpha}{1-\rho}$, $\forall n\in\mathbb{Z}$. The first term in the right-hand side of inequality (4.71) is finite by Young's inequality, and the fact that $B\in l^2$. Thus S is also Hilbert–Schmidt. ∎

Let B_1 be as in (4.30). Also for $\kappa > 0$, $\rho = e^{-\kappa}$, define the set $B_{2,\kappa}$ by

$$B_{2,\kappa} = \left\{ -4\delta \sin^2\frac{k}{2}\left(2\omega\frac{1-\rho^2}{1-2\rho\cos k+\rho^2} + 4\delta\sin^2\frac{k}{2}\right) : \quad k \in \mathbb{R}) \right\}. \tag{4.72}$$

THEOREM 4.2. *The essential spectrum of the operators $\overline{\mathcal{L}}_2$ and $\mathcal{L}_{\delta,\kappa}$ in Y includes the set of $\lambda \in \mathbb{C}$ satisfying $\lambda^2 \in B_1 \cup B_{2,\kappa}$.*

The two scenarios are suggested by the numerical results of [1], Figures 3, 4. These show calculations with finite shelf-type breathers with support of about $m = 40$ sites (in the limit $\delta = 0$), but we start to see the accumulation of eigenvalues in the regions indicated by $\lambda^2 \in B_1 \cup B_{2,\kappa}$.

Proof. By Lemma 4.5 it is enough to examine the essential spectrum of $\overline{\mathcal{L}}_2$. We compute approximate eigenfunctions of $\overline{\mathcal{L}}_2$, using the constant coefficient operators $\overline{\mathcal{L}}_l$, $\overline{\mathcal{L}}_r$ in $X \times X$ defined in the following way. Consider operators A^0, M, h in X with entries in the standard basis $M(n,m) = e^{-\kappa|n-m|}$, $A^0(n,m) = \frac{\omega}{2}\delta_{n,m}$, n, $m \in \mathbb{Z}$. Let $h(n,m) = \delta_{n,m}$ if n, $m \geq 0$, $h(n,m) = 0$ otherwise. Then let

$$\overline{\mathcal{L}}_l = \begin{bmatrix} 0 & -\omega I - \delta\Delta + 2A^0 \\ \omega I + \delta\Delta - 2A^0 - 2\omega M & 0 \end{bmatrix}, \quad \overline{\mathcal{L}}_r = \begin{bmatrix} 0 & -\omega I - \delta\Delta \\ \omega I + \delta\Delta & 0 \end{bmatrix}. \tag{4.73}$$

In comparison, the definition of $\overline{\mathcal{L}}_2$ implies

$$\overline{\mathcal{L}}_2 = \begin{bmatrix} 0 & -\omega I - \delta\Delta + 2hA^0 \\ \omega I + \delta\Delta - 2hA^0 - 2\omega hM & 0 \end{bmatrix}. \tag{4.74}$$

Let $k \in \mathbb{R}$, and consider $z^k \in l^\infty(\mathbb{Z},\mathbb{C})$ of the form

$$z_n^k = [a_n e^{ikn} + a_n^* e^{-ikn}, b_n e^{ikn} + b_n^* e^{-ikn}]^T. \tag{4.75}$$

Then

$$\overline{\mathcal{L}}_r z^k = \lambda z^k, \quad \forall k \in \mathbb{Z}, \tag{4.76}$$

implies

$$\lambda^2 = -(\omega - 4\delta\sin^2\frac{k}{2})^2. \tag{4.77}$$

Also,

$$\overline{\mathcal{L}}_l z^k = \lambda z^k, \quad \forall k \in \mathbb{Z}, \tag{4.78}$$

implies

$$\lambda^2 = -4\delta\sin^2\frac{k}{2}\left(2\omega\frac{1-\rho^2}{1-2\rho\cos k+\rho^2} + 4\delta\sin^2\frac{k}{2}\right). \tag{4.79}$$

Let $k \in \mathbb{R}$, and $\lambda^2 = \lambda^2(k)$ as in equation (4.77). For $\mu \geq 5$ integer, define $v_\mu \in X_c \times X_c$ by

$$v_\mu(n) = \begin{cases} \frac{1}{\sqrt{2\mu}} e^{ikn}[1,1]^T, \text{ if} & n \in \{1,\ldots,\mu\} \\ 0, & \text{if } n \in \mathbb{Z}\setminus\{1,\ldots,\mu\}. \end{cases} \tag{4.80}$$

Then $||v_\mu||_{Y_c}=1$, $\forall\mu\geq 5$. Note also that $A^0 v_\mu=0$, and $Mv_\mu=0$, $\forall\mu\geq 5$. Letting $[q_\mu,p_\mu]^T=v_\mu$, and using equation (4.74) we then have that

$$||(\overline{\mathcal{L}}_2-\lambda)v_m||^2_{Y_c}=\sum_{n\in\mathbb{Z}}\left(|[(-\omega I-\delta\Delta)p_\mu-\lambda q_\mu]_n|^2+|[(\omega I+\delta\Delta)q_\mu-\lambda p_\mu]_n|^2\right). \quad (4.81)$$

By equations (4.76) and (4.77) the only nonvanishing terms in the sum (4.81) are the ones corresponding to the four indices $n=0$, 1, μ, $\mu+1$. This holds for any $\mu\geq 5$. Each term is proportional to μ^{-1}, and we easily see that

$$||\overline{\mathcal{L}}_2 v_\mu-\lambda v_m||^2_{Y_c}\leq\frac{4}{\mu}(2|\omega|+7|\delta|)^2\to 0, \quad \text{as} \quad \mu\to\infty. \quad (4.82)$$

Thus $\lambda(k)$, with λ^2 as in equation (4.77), belongs to the essential spectrum of $\overline{\mathcal{L}}_2$. Varying k over $\mathbb{R}$ we have that all $\lambda^2\in B_1$ belong to $\sigma_{ess}(\overline{\mathcal{L}}_2)$.

Let $k\in\mathbb{R}$, and $\lambda^2=\lambda^2(k)$ as in equation (4.79). For $\mu\geq 5$ integer, define $w_\mu\in X_c\times X_c$ by

$$w_\mu(n)=\begin{cases}\frac{1}{\sqrt{2(\mu+1)}}e^{ikn}[1,1]^T, \text{ if} & n\in\{-\mu,\dots,0\}\\ 0, & \text{if } n\in\mathbb{Z}\setminus\{-\mu,\dots,0\}\end{cases} \quad (4.83)$$

We have $||w_\mu||_{Y_c}=1$, $\forall\mu\geq 5$.

We examine first the components $(\overline{\mathcal{L}}_2 w_\mu-\lambda w_\mu)_n$, with $n>0$. We have $(A^0 w_\mu)_n=0$, and $(Mw_\mu)_n=0$, $\forall n>0$. Letting $[q_\mu,p_\mu]^T=w_\mu$, $(\Delta q_\mu)_n$, $(\Delta p_\mu)_n$ vanish for all $n>1$. We then compute

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(|[(-\omega I-\delta\Delta+2hA^0)p_\mu-\lambda q_\mu]_n|^2+\right.\\ &\left.\quad+|[(\omega I+\delta\Delta-2hA^0-2\omega hM)q_\mu-\lambda p_\mu]_n|^2\right)=\frac{1}{\mu+1},\end{aligned} \quad (4.84)$$

for any $\mu\geq 5$.

Examining the components $(\tilde{\mathcal{L}}w_\mu-\lambda w_\mu)_n$ with $n\leq 0$, we have that by equations (4.78) and (4.79)

$$[(-\omega I-\delta\Delta+2hA^0)p_\mu-\lambda q_\mu]_n=0, \quad \forall n\in\{-\mu+1,\dots,-1\}, \quad \text{and} \quad \forall n\leq-\mu-2. \quad (4.85)$$

For $n=0$, $-\mu$, $-\mu-1$ these terms are proportional to $(\mu+1)^{-1/2}$, and we compute

$$\sum_{n=-\infty}^{0}\left|[(-\omega I-\delta\Delta+2hA^0)p_\mu-\lambda q_\mu]_n\right|^2\leq\frac{1}{\mu+1}[(3|\delta|+|\lambda|)^2+1] \quad (4.86)$$

$$\leq\frac{1}{\mu+1}\left[2|\delta|^2+4|\delta|\left(\frac{2|\omega|}{1-\rho}+4|\delta|\right)+1\right], \quad (4.87)$$

where $|\lambda|^2$ was bounded using equation (4.79). The above holds for any $\mu\geq 5$.

By $n\leq 0$ and the definition of hA^0 we also have

$$[(\omega I+\delta\Delta-2hA^0-2\omega hM)q_\mu-\lambda p_\mu]_n=[(\delta\Delta-2\omega hM)q_\mu-\lambda p_\mu]_n. \quad (4.88)$$

Consider $n=-\mu-r$, $r\geq 2$. Then $[\Delta q_\mu]_n=\lambda p_\mu=0$. Also

$$[hMq_\mu]_n=\frac{1}{\sqrt{2(\mu+1)}}\sum_{m\in\{-\mu,\dots,0\}}\rho^{|-\mu-r-m|}e^{ikm} \quad (4.89)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2(\mu+1)}} \rho^r \rho^\mu \sum_{m \in \{-\mu,\ldots,0\}} \rho^m e^{ikm} \\
&= \frac{1}{\sqrt{2(\mu+1)}} \rho^{r-1} \frac{\rho^{\mu+1} - e^{-i(\mu+1)k}}{1-\rho}. \qquad (4.90)
\end{aligned}
$$

By equations (4.88) and (4.90) we have

$$
\begin{aligned}
\sum_{n=-\infty}^{-\mu-2} |[(\omega I + \delta\Delta - 2hA^0 - 2\omega h M) q_\mu - \lambda p_\mu]_n|^2 &= \sum_{n=-\infty}^{-\mu-2} |2\omega h M q_\mu]_n|^2 \qquad (4.91) \\
&= \sum_{r=2}^{\infty} \frac{2\omega^2 \rho^{2(r-1)}}{(\mu+1)} \frac{(\rho^{\mu+1} - e^{-i(\mu+1)k})^2}{(1-\rho)^2} \\
&= \frac{2\omega^2 (\rho^{\mu+1} - e^{-i(\mu+1)k})^2 \rho^2}{(\mu+1)(1-\rho)^2(1-\rho^2)}, \qquad (4.92)
\end{aligned}
$$

for any $\mu \geq 5$.

For $n = -\mu - 1$, we can use equation (4.90) with $r = 1$. We have

$$
[-2\omega h M) q_\mu]_{-\mu-1} = -2\omega \frac{1}{\sqrt{2(\mu+1)}} \frac{\rho^{\mu+1} - e^{-i(\mu+1)k}}{1-\rho}, \qquad (4.93)
$$

and therefore

$$
|[(\omega I + \delta\Delta - 2hA^0 - 2\omega h M) q_\mu - \lambda p_\mu]_{-\mu-1}|^2 = \frac{1}{2(\mu+1)} \left(2|\omega| \left| \frac{\rho^{\mu+1} - e^{-i(\mu+1)k}}{1-\rho} \right| + |\delta| \right), \qquad (4.94)
$$

for any $\mu \geq 5$.

We now consider $n \in \{-\mu+1, \ldots, 1\}$. To estimate

$$
T_n = [(\omega I + \delta\Delta - 2hA^0 - 2\omega h M) q_\mu - \lambda p_\mu]_n = [(\delta\Delta - 2\omega h M) q_\mu - \lambda p_\mu]_n, \qquad (4.95)
$$

we use the fact that w_μ satisfies equation (4.78), that is

$$
[(\delta\Delta - 2\omega M) q_\mu - \lambda p_\mu]_n = 0, \quad \forall n \in \{-\mu+1, \ldots, 1\}, \qquad (4.96)
$$

and equation (4.95) to write

$$
T_n = [-2\omega(hM - M) q_\mu]_n, \quad \forall n \in \{-\mu+1, \ldots, 1\}. \qquad (4.97)
$$

By the definitions of h, and M we have

$$
\begin{aligned}
[hMq_\mu]_n &= (2\mu+2)^{-1/2} \sum_{m \in \{-\mu,\ldots,0\}} \rho^{|m-n|} e^{ikn} \qquad (4.98) \\
&= (2\mu+2)^{-1/2} \left(\sum_{m \in \{n,\ldots,0\}} \rho^{m-n} e^{ikn} + \sum_{m \in \{-\mu,\ldots,n-1\}} \rho^{n-m} e^{ikn} \right) \qquad (4.99) \\
&= (2\mu+2)^{-1/2} \left(\sum_{m=n}^{0} \rho^{m-n} e^{ik(m-n)} + \sum_{m=-\mu}^{n-1} \rho^{n-m} e^{-ik(n-m)} \right) e^{ikn} \qquad (4.100) \\
&= (2\mu+2)^{-1/2} \left(\sum_{\lambda=0}^{|n|} \rho^\lambda e^{ik\lambda} + \sum_{\lambda=1}^{\mu-|n|} \rho^\lambda e^{-ik\lambda} \right) e^{ikn}, \qquad (4.101)
\end{aligned}
$$

and

$$[Mq_\mu]_n = (2\mu+2)^{-1/2}\left(\sum_{\lambda=0}^{\infty}\rho^\lambda e^{ik\lambda} + \sum_{\lambda=1}^{\infty}\rho^\lambda e^{-ik\lambda}\right)e^{ikn}. \tag{4.102}$$

By equations (4.97), (4.101), and (4.102) we then have

$$T_n = \frac{2\omega}{\sqrt{2(\mu+1)}}\left(\sum_{\lambda=|n|+1}^{\infty}\rho^\lambda e^{ik\lambda} + \sum_{\lambda=|\mu|-|n|+1}^{\infty}\rho^\lambda e^{-ik\lambda}\right)e^{ikn} \tag{4.103}$$

$$= \frac{2\omega}{\sqrt{2(\mu+1)}}\left(\frac{\rho^{|n|+1}e^{ik(|n|+1)}}{1-\rho e^{ik}} + \frac{\rho^{\mu-|n|+1}e^{-ik(\mu-|n|+1)}}{1-\rho e^{-ik}}\right)e^{ikn}. \tag{4.104}$$

Then

$$\sum_{n=-\mu+1}^{1}|T_n|^2 = \frac{2\omega}{\mu+1}\sum_{n=-\mu+1}^{1}\left(\frac{\rho^{2(|n|+1)}}{|1-\rho e^{ik}|^2} + \frac{\rho^{2(\mu-|n|+1)}}{|1-\rho e^{-ik}|^2} + 2\mathrm{Re}\frac{\rho^{\mu+2}e^{ik(\mu+2)}}{(1-\rho e^{ik})(1-\rho e^{-ik})}\right). \tag{4.105}$$

The squared modulus terms are bounded by geometric series, e.g.

$$|1-\rho e^{ik}|^{-2}\sum_{n\in\{-\mu+1,\dots,1\}}\rho^{2(|n|+1)} \le \frac{1}{(1-\rho)^2}\frac{1}{1-\rho^2}, \tag{4.106}$$

similarly for the second term. The sum of cross terms is bounded as

$$\sum_{n\in\{-\mu+1,\dots,1\}} 2\mathrm{Re}(\rho^{\mu+2})e^{ik(\mu+2)}(1-\rho e^{ik})^{-1}(1-\rho e^{-ik})^{-1} \le \frac{2(\mu-1)\rho^{\mu+2}}{(1-\rho)^2}. \tag{4.107}$$

Then equation (4.105) yields

$$\sum_{n\in\{-\mu+1,\dots,1\}}|T_n|^2 \le \frac{2\omega}{\mu+1}\left[\frac{2}{1-\rho^2} + 2(\mu-1)\rho^{\mu+2}\frac{1}{(1-\rho)^2}\right]. \tag{4.108}$$

This holds for all $\mu \ge 5$.

We finally consider the two sites $n=-\mu$, and $n=0$. At $n=-\mu$ we have

$$[hMq_\mu]_{-\mu} = \frac{1}{\sqrt{2(\mu+1)}}\sum_{m\in\{-\mu,\dots,0\}}\rho^{|-\mu-m|}e^{ikm} \tag{4.109}$$

$$= \frac{1}{\sqrt{2(\mu+1)}}e^{-ik\mu}\frac{1-(\rho e^{ik})^{\mu+1}}{1-\rho e^{ik}}, \tag{4.110}$$

therefore

$$|[(\delta\Delta - 2\omega hM)q_\mu - \lambda p_\mu]_{-\mu}|^2 \le \frac{1}{2(\mu+1)}\left(3|\delta| + |\lambda| + 2|\omega|\frac{1}{1-\rho}\right), \tag{4.111}$$

with $|\lambda|$ bounded in terms of $|\delta|$, $|\omega|$ as in inequality (4.86). At $n=0$, $[hMq_\mu]_0$ is similarly

$$[hMq_\mu]_0 = \frac{1}{\sqrt{2(\mu+1)}}\sum_{m\in\{-\mu,\dots,0\}}\rho^{|-m|}e^{ikm} = \frac{1}{\sqrt{2(\mu+1)}}\frac{1-(\rho e^{-ik})^{\mu+1}}{1-\rho e^{-ik}}, \tag{4.112}$$

and

$$|[(\delta\Delta - 2\omega h M)q_\mu - \lambda p_\mu]_0|^2 \leq \frac{1}{2(\mu+1)}\left(3|\delta| + |\lambda| + 2|\omega|\frac{1}{1-\rho}\right), \tag{4.113}$$

with $|\lambda|$ bounded in terms of $|\delta|$, $|\omega|$ as in inequality (4.86). As before, inequalities (4.111) and (4.113) are valid for any $\mu \geq 5$.

Collecting the results (4.84), (4.86), (4.94), (4.108), (4.111), and (4.113) we see that there exists a C that depends on $|\omega|$, $|\delta|$,and κ for which

$$||\overline{\mathcal{L}}_2\mu - \lambda w_m||^2_{Y_c} \leq \frac{C}{\mu+1} \to 0, \quad \text{as} \quad \mu \to \infty, \tag{4.114}$$

thus $\lambda(k)$, λ^2 as in (4.79), belongs to the essential spectrum of $\overline{\mathcal{L}}_2$. Varying over $k \in \mathbb{R}$ we have that all $\lambda^2 \in B_{2,\kappa}$ belong to $\sigma_{ess}(\overline{\mathcal{L}}_2)$. ◻

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