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Near-monochromatic water waves on the sphere

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Abstract

We study analytically and numerically a class of traveling and standing waves in a model of weakly non-linear gravity water waves on the sphere. These waves are 'near-monochromatic' in space, i.e. their amplitude consists of one spherical harmonic plus small corrections, and we see numerically that they retain this property for long time. A main feature of the model we consider is that it possesses a Hamiltonian structure. This structure is preserved by our numerical implementation, and we use formal and rigorous arguments from classical perturbation theory to understand the numerical observations. ©1999 Elsevier Science B.V. All rights reserved.

1. Introduction

We present an analytical and numerical study of a class of weakly non-linear traveling and standing waves in a model of surface water waves on the sphere. In earlier work on a somewhat more general model that includes the present one we have established analytically the existence of families of approximate periodic orbits that are "near-monochromatic" in space, i.e. the wave amplitude consists of a spherical harmonic plus small corrections (see [1]). The present paper concerns the stability of such motions, emphasizing their spatial features and the transfer of energy between the various spherical harmonic modes. Our main numerical observation is that trajectories starting from initial conditions consisting of one spherical harmonic plus small terms exhibit small transfer of energy to other modes for long times. In addition to supporting this claim we also propose a partial theoretical explanation of the observed behavior.

Our study is based on a Hamiltonian model for free surface potential flow of a fluid layer surrounding a gravitating sphere. The theory was developed in [1] and is a generalization to the spherical case of the Hamiltonian theory of water waves of Zakharov [2] (see also [3]). The equations of motion are integro-differential and in our derivation of approximate Hamiltonians for small amplitude oscillations we have incorporated work of [4,5] on the non-local operators appearing in the problem. An advantage of the Hamiltonian formalism is that it yields Hamiltonian Galerkin

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truncations of the equations of motion. Thus we solve numerically a finite dimensional Hamiltonian system, and we use methods from canonical perturbation theory to understand the numerical observations. In the simulations we have considered a relatively thin fluid layer for which the cubic Hamiltonian provides a reasonable approximation. At the same time we are sufficiently far from the regime where cubic resonant interactions dominate the dynamics, and our theory is based on the analysis of quartic resonant interaction terms.

The theoretical tool we will use is a Birkhoff normal form calculation. We see that the quartic normal form Hamiltonian of the system we integrated numerically possesses several quantities ('asymptotic invariants') that evolve slowly and control the transfer of energy between the different spherical harmonic modes for long time. In the case of axisymmetric oscillations the quartic normal form is completely integrable and the existence of asymptotic invariants gives a complete explanation of the numerical results. For general initial conditions, the asymptotic invariants leave only a few modes to which a near-monochromatic oscillation may decay. For certain (non-axisymmetric) initial conditions a heuristic argument suggests that decay to even these few modes must be slow.

Note that although our arguments are detailed for the particular model we have studied numerically, they also apply to larger finite dimensional truncations of the equations of motion, and also to models with higher order non-linearity. However, we cannot at present extend our theory to the infinite dimensional case.

The Hamiltonian formalism and Birkhoff normal forms for water wave problems have been considered by several authors. Of special interest are limiting cases where quartic normal forms are completely integrable, see e.g. [6,7] on 2-D deep water waves. Such stronger results rely on detailed information on the non-linearities, and since they concern different geometries, their connection with the spherical problem is not clear. We therefore discuss them briefly in the conclusion of this paper. Also, there is an extensive literature on the drift of the actions in near-integrable Hamiltonian systems. These ideas should be particularly relevant to axisymmetric motions (see especially [8] and references therein), and they suggest that other motions, e.g. with many spherical harmonic modes should also be stable for long time. One difficulty in applying such results to our problem directly is that we cannot control the distance from the quartic resonances so well. Also there may be regions of phase space where the quartic normal form is degenerate. The situation for general initial conditions is more complicated since the normal form is not integrable, and this is where our numerical results and the theory are more interesting and novel.

The Hamiltonian water wave model and our numerical implementation are described in Section 1. In Section 2 we present the results of the numerical simulations. In Section 4 we describe normal form arguments that explain the numerical observations, and discuss at some length the physical validity of the model we studied numerically as well as extensions of the theory to larger models. In Section 5 we prove rigorous versions of some of the arguments of Section 4, and in Section 6 we summarize and discuss our results.

2. Hamiltonian formulation and numerical implementation

We consider a fluid layer of thickness h surrounding a gravitating sphere of radius b. Using polar coordinates r = radius, ϑ = polar, φ = azimuth, the outer (or 'free') surface of the layer will be the set of points with radius $r(\vartheta,\varphi)=\rho+\eta(\vartheta,\varphi)$, with $\rho=b+h$. We are interested in the oscillations of the free surface around the quiescent state with $\eta(\vartheta,\varphi)=0$, and we assume that the flow inside the layer is potential. Denoting the hydrodynamic potential by ϕ, Euler's equations for free surface potential flow take the form (see e.g. [9])

$$\frac{\partial\eta}{\partial t}=\frac{\partial\phi}{\partial r}-\frac{1}{r^2}\frac{\partial\phi}{\partial\vartheta}\frac{\partial\eta}{\partial\vartheta}-\frac{1}{r^2\sin^2\vartheta}\frac{\partial\phi}{\partial\varphi}\frac{\partial\eta}{\partial\varphi}, \tag{1}$$

and

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}|\nabla\phi|^2 + \frac{1}{\rho+\eta}, \tag{2}$$

at the free surface,

$$\Delta\phi = 0 \tag{3}$$

in the region occupied by the fluid, and

$$\frac{\partial \phi}{\partial r} = 0 \tag{4}$$

at the bottom $r = b$. Note that the wave amplitude $\eta(\vartheta, \varphi)$ and the surface potential $\Phi(\varphi, \vartheta) = \phi(\varphi, \vartheta, \rho + \eta(\vartheta, \varphi))$ determine the hydrodynamic potential ϕ inside the layer uniquely at each instant.

The above equations can also be written as a Hamiltonian system (the Hamiltonian formulation for the sphere was developed in [1], here we only outline the relevant results). The canonical variables are the wave amplitude η, and the hydrodynamic potential at the surface Φ, while the Hamiltonian H is the total energy (kinetic + potential) of the system. The kinetic energy part of the Hamiltonian can be formally expanded in powers of the wave amplitude η, and we may write $H = \sum_{j=0}^{\infty} H_j$, with H_j of order $j+2$ in the canonical variables. To rewrite (1)–(4) as Hamilton's equations it is convenient to use the variables $\tilde{\eta}$ and $\tilde{\Phi}$, defined by $\tilde{\eta} = \eta\,(1 + \eta/2\rho)$, $\tilde{\Phi} = \Phi\,(1 + \eta/\rho)$. From now on all quantities will be expressed in terms of $\tilde{\eta}$ and $\tilde{\Phi}$ and we drop the tilde from the notation. Note that the H_j are modified so that they still consist of the terms of order $j + 2$ in the new variables. Also, we expand η and Φ in spherical harmonics $Y_\gamma(\vartheta, \varphi)$, with $\gamma = [l, m], l = 1, 2, \dots, m = -l, \dots, l$ (see e.g. [10]), i.e. $\eta = \sum_\gamma \eta_\gamma Y_\gamma$, $\Phi = \sum_\gamma \Phi_\gamma Y_\gamma$. Then the equations of motion for each mode η_γ, Φ_γ are

$$\dot{\eta}_\gamma = \frac{\partial H}{\partial \Phi_\gamma^*}, \quad \dot{\Phi}_\gamma = -\frac{\partial H}{\partial \eta_\gamma^*}, \tag{5}$$

and the quadratic and cubic parts of the Hamiltonian are

$$H_0 = \frac{\rho^2}{2}\sum_\gamma \left(\frac{u'_\gamma(\rho)}{u_\gamma(\rho)}\Phi_\gamma\Phi_\gamma^* + \eta_\gamma\eta_\gamma^*\right), \quad \text{with} \quad u_\gamma(r) = (l+1)\left(\frac{r}{b}\right)^l + l\left(\frac{b}{r}\right)^{l+1}, \tag{6}$$

$$H_1 = \sum_{\gamma_1,\gamma_2,\gamma_3} I_{\gamma_1,\gamma_2,\gamma_3}\Phi_{\gamma_1}\Phi_{\gamma_2}\eta_{\gamma_3}, \tag{7}$$

with

$$I_{\gamma_1,\gamma_2,\gamma_3} = \frac{\rho^2}{2}\left(\frac{u''_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} - \frac{u'_{\gamma_1}(\rho)}{u_{\gamma_1}(\rho)}\frac{u'_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)}\right)\int_{S^2} Y_{\gamma_1}Y_{\gamma_2}Y_{\gamma_3} + \frac{1}{2}\int_{S^2} Y_{\gamma_1}\nabla Y_{\gamma_2}\cdot\nabla Y_{\gamma_3}.$$

The quartic part of the Hamiltonian is computed explicitly in [1], where we also give a recursive formula for higher order non-linearities. According to the dimensional analysis of [1] the Hamiltonian can be written as $H = \sum_{j=0}^{\infty}\epsilon^j H_j$, with ϵ the ratio of a typical wave amplitude to the depth h. Moreover, in each H_j, $j > 0$ we can factor out a term β^j, where $\beta = h/b$. In the present work we have fixed $b = 1.0$ and $h = 0.2$, and we have considered the cubic Hamiltonian $H = H_0 + \epsilon H_1$. Since β is small the cubic Hamiltonian gives plausible model for wave motions with horizontal length scales that are comparable to the radius of the sphere. A more complete justification of this approximation, and a discussion of the effects of higher order non-linear terms will be given at the end of Section 4.

To solve the initial value problem numerically, we consider Galerkin truncations of Eqs. (5), i.e. the summations in the Hamiltonian of (6) and (7) are replaced by summations over a finite set of spherical harmonic mode indices. In our observations we have used a Galerkin truncation that includes all spherical harmonic modes with $l \leq l_{\max} = 8$. Taking into account the real and imaginary parts of η_γ, Φ_γ and the reality condition $\eta_{[l,-m]} = \eta^*_{[l,m]}$, $\Phi_{[l,-m]} = \Phi^*_{[l,m]}$ we then have a Hamiltonian system in $\mathbb{R}^{2N}$, with $N = 80 (N = l^2_{\max} + 2l_{\max})$. We may also express the equations of motion and the Hamiltonian (5)–(7) using the variables a_γ, a^*_γ, defined by

$$\eta_\gamma = \frac{\sqrt{2\omega_\gamma}}{2}(a_\gamma + a^*_{\bar{\gamma}}), \quad \Phi_\gamma = -i\frac{\sqrt{2}}{2\sqrt{\omega_\gamma}}(a_\gamma - a^*_{\bar{\gamma}}) \quad \text{with} \quad \omega^2_\gamma = \frac{u'_\gamma(\rho)}{u_\gamma(\rho)} \tag{8}$$

i.e. ω_γ is the dispersion relation. Letting $q_\gamma = \mathrm{Re}(a_\gamma)$ and $p_\gamma = Im(a_\gamma)$, Hamilton's equations (5) become

$$\dot{q}_\gamma = \frac{1}{2}\frac{\partial H}{\partial p_\gamma}, \quad \dot{p}_\gamma = -\frac{1}{2}\frac{\partial H}{\partial q_\gamma}. \tag{9}$$

Equation (9), with the Hamiltonian $H = H_0 + \epsilon H_1$, and the truncation and parameters b and h above, are the equations we integrate numerically. To integrate the equations we used the Fortran package LSODE, available through netlib (see [11]). The integration algorithm we chose is a predictor-corrector multi-step method, where the prediction step is performed by an (explicit) Adams–Bashforth formula, and the correction step is done by an (implicit) Adams–Moulton formula (see [12]). The choice of step-size and order is performed automatically during the integration so that the computed values of q_γ, p_γ are within a user-specified local error tolerance. The estimation of the local error, and the procedure for choosing the step-size and the order are described in [13], p. 155 (see also [14]). Tests of the accuracy of the numerical integration are reported in [15].

3. Numerical observations

We are interested in the evolution from initial conditions where one spherical harmonic mode a_Γ has initial amplitude O(1) and all other modes a_γ with $\gamma \neq \Gamma$ have initial amplitude $\ll$O(1). We refer to such initial conditions as 'near-monochromatic'. Our basic observation is that numerical trajectories starting from near-monochromatic initial conditions remain near-monochromatic for long time.

In presenting the numerical evidence for this behavior we will consider separately two cases, axisymmetric oscillations and general oscillations. Axisymmetric oscillations are trajectories belonging to the vector space $V_A = \text{span}\,\{a_\gamma : \gamma = [l, 0]\}$. The invariance of $V_A \subset \mathbb{R}^{2N}$ under the flow of Hamilton's equations (9) follows from the fact that the coefficients $I_{\gamma_1\gamma_2\gamma_3}$ of the cubic Hamiltonian H_1 in (7) vanish if $m_1 + m_2 + m_3 \neq 0$. Although the procedure we followed and the numerical results are very similar for both axisymmetric and general initial conditions, the distinction is useful in the theoretical discussion of the next section. Also note that in all the numerical experiments described below, the motion on the plane a_Γ is (roughly) a sinusoidal oscillation with an amplitude modulation. Thus, in space, the motions are close to those of the linear equations with $a_\gamma(0) = 0$ for $\gamma \neq \Gamma$, and can be characterized as traveling waves (rotating around the north-south axis) for $\Gamma = [l, m]$ with $m \neq 0$, and standing waves in the axisymmetric case.

In a first set of measurements, summarized below, the goal was to quantify our observation that near-monochromatic solutions do not decay for long time. The emphasis was on trying a large number of initial conditions. The reliability of these numerical observations will be discussed afterwards.

3.1. Axisymmetric motions

We have considered a set of axisymmetric initial conditions of the form $|a_\Gamma(0)| = 1.0$, and $|a_\gamma(0)| \leq 5 \times 10^{-2}$ for $\gamma \neq \Gamma$. The index Γ in this set ranges from $\Gamma = [1, 0]$ to $\Gamma = [5, 0]$. The (non-linearity) parameter ϵ was varied in the interval $[10^{-4}, 10^{-2}]$. Note that for ϵ around 5×10^{-2}, depending on the initial condition, trajectories start to become unbounded in finite time. Since we are considering trajectories starting from different initial conditions, we want to avoid these "blow-ups", (which, incidentally, are not due to numerical instability, see [15]). The observation time T is 250 (i.e. 50–80 periods of the linear motion, depending on the l_Γ). The local error tolerance W (see [13] p.155) is 10^{-9}. For such W, energy is conserved to $10^{-5} - 10^{-5}$ over $T = 250$ (for total energy ~ 1.0). For each mode a_γ we observe the amplitude variation $\Delta A_\gamma(T)$, defined by

$$\Delta A_\gamma(T) \equiv \sup_{t\in[0,T]} |A_\gamma(t) - A_\gamma(0)|, \quad A_\gamma(t) = \sqrt{a_\gamma(t) a^*_\gamma(t)}. \tag{10}$$

Our main observations are as follows.

1. The mode a_Γ does not decay. The range of observed $\Delta A_\Gamma(T)$ for $T = 250$ is: for $\epsilon = 10^{-4}$, $\Delta A_\Gamma(T) \in [10^{-5}, 10^{-4}]$; for $\epsilon = 10^{-3}$, $\Delta A_\Gamma(T) \in [10^{-4}, 10^{-3}]$; for $\epsilon = 10^{-2}$, $\Delta A_\Gamma(T) \in [10^{-3}, 10^{-2}]$. For a given initial condition, $\Delta A_\Gamma(T)$ is roughly (but not exactly) proportional to ϵ.
2. The transfer of energy (or amplitude) to other modes is limited. If we let $a_{\tilde\gamma}$ be the mode with the largest amplitude variation $\Delta A_{\tilde\gamma}(T)$ over T, we have seen that for $\epsilon = 10^{-4}$, $\Delta A_{\tilde\gamma}(T) \in [10^{-4}, 10^{-3}]$; for $\epsilon = 10^{-3}$, $\Delta A_{\tilde\gamma}(T) \in [10^{-3}, 10^{-2}]$; for $\epsilon = 10^{-2}$, $\Delta A_{\tilde\gamma}(T) \in [10^{-2}, 1.7 \times 10^{-1}]$. Other modes a_γ, with $\gamma \neq \Gamma, \tilde\gamma$, may have $\Delta A_\gamma(T)$ with same order of magnitude as $\Delta A_{\tilde\gamma}(T)$. Clearly for $\epsilon = 10^{-2}$ the energy transfer to other modes is appreciable, but still the mode Γ dominates. We generally observe that, for given initial conditions, the $\Delta A_\gamma(T)$ are roughly (but not exactly) proportional to ϵ for the various modes a_γ. Also, in all experiments we see that $A_\gamma(t)$ oscillates so that in general $\Delta A_\gamma(T)$ is attained quite early, by $T \sim 50$. This fact suggests that the measured drift of the amplitudes should give information for much longer intervals.

3.2. General motions

We consider initial conditions of the form $|a_\Gamma(0)| = 1.0$ and $|a_\gamma(0)| \leq 10^{-2}$, for $\gamma \neq \Gamma$. The index Γ ranges over the modes with $l \leq 4$. The error tolerance W, the range of ϵ and the observation time T are the ones used for the axisymmetric case. Again, we measure the quantity $\Delta A_\gamma(T)$ defined as in (10). Our main observations are summarized below.

1. The mode Γ does not decay. Overall we have for $\epsilon = 10^{-4}$, $\Delta A_\Gamma(T) \in [10^{-6}, 10^{-5}]$; for $\epsilon = 10^{-3}$, $\Delta A_\Gamma(T) \sim 10^{-4}$; for $\epsilon = 10^{-2}$, $\Delta A_\Gamma(T) \sim 10^{-2}$.
2. Transfer of energy to other modes is limited. We generally observe that for $\epsilon = 10^{-4}$, $\Delta A_{\tilde\gamma}(T) \in [10^{-4}, 10^{-3}]$; for $\epsilon = 10^{-3}$, $\Delta A_{\tilde\gamma}(T) \in [10^{-3}, 10^{-2}]$; for $\epsilon = 10^{-2}$, $\Delta A_{\tilde\gamma}(T) \in [10^{-2}, 1.5 \times 10^{-2}]$. Again, the transfer of amplitude to other modes can be appreciable as ϵ increases, but the mode Γ dominates, and also the $\Delta A_\gamma(T)$ is attained by $T \sim 50$.

In a second set of experiments we assess the accuracy and reliability of the above numerical results. Our strategy is to consider a smaller set of initial conditions and integrate numerically for longer time and with smaller local error tolerance W. In particular, we repeated the first set of experiments with the same initial conditions, but for $\epsilon = 0.001$ only. The integration time is now $4T = 1000$ and $W = 10^{-15}$ and 10^{-9}. We observe that:

1. Generally, the $\Delta A_\gamma(T)$ obtained with $W = 10^{-15}$ and 10^{-9} agree to at least 1–3% (usually to higher accuracy). Similar agreement is observed between the $\Delta A_\gamma(4T)$ obtained with $W = 10^{-15}$ and 10^{-9}.
2. Generally, $\Delta A_\gamma(T)$ and $\Delta A_\gamma(4T)$ are very close, to within 1–3%, for both $W = 10^{-15}$ and 10^{-9}. Typical data indicating (1) and (2) are in Tables 1 and 2 where the dominant modes a_Γ are $a_{[3,2]}$ and $a_{[4,3]}$ respectively. The second line on each table shows the largest $\Delta A_\gamma(T)$, i.e. $\Delta A_{\tilde\gamma}(T)$, and the other lines show other modes with the highest amplitude variation.

Table 1
$\Gamma = [3, 2]$. W is local error tolerance.

γ	$W = 10^{-9}$ $\Delta A_\gamma(T)$[a]	$W = 10^{-9}$ $\Delta A_\gamma(4T)$	$W = 10^{-15}$ $\Delta A_\gamma(T)$	$W = 10^{-15}$ $\Delta A_\gamma(4T)$
[3, 2]	1.4000×10^{-4}	1.4000×10^{-4}	1.3000×10^{-4}	1.3000×10^{-4}
[6, 4]	1.1056×10^{-2}	1.1056×10^{-2}	1.1056×10^{-2}	1.1056×10^{-2}
[4, −4]	1.4162×10^{-3}	1.4162×10^{-3}	1.4162×10^{-3}	1.4162×10^{-3}
[6, 0]	1.2863×10^{-3}	1.2863×10^{-3}	1.2863×10^{-3}	1.2863×10^{-3}

[a]Time $T = 250$.

Table 2
$\Gamma = [4, 3]$. W is local error tolerance.

γ	$W = 10^{-9}$ $\Delta A_\gamma(T)$[a]	$W = 10^{-9}$ $\Delta A_\gamma(4T)$	$W = 10^{-15}$ $\Delta A_\gamma(T)$	$W = 10^{-15}$ $\Delta A_\gamma(4T)$
[4, 3]	2.5000×10^{-4}	2.8000×10^{-4}	2.4000×10^{-4}	2.4000×10^{-4}
[8, 6]	1.4466×10^{-2}	1.4466×10^{-2}	1.4465×10^{-2}	1.4465×10^{-2}
[6, 6]	4.3552×10^{-3}	4.3552×10^{-3}	4.3552×10^{-3}	4.3552×10^{-3}
[8, −6]	2.8724×10^{-3}	2.8724×10^{-3}	2.8724×10^{-3}	2.8724×10^{-3}

[a]Time $T = 250$.

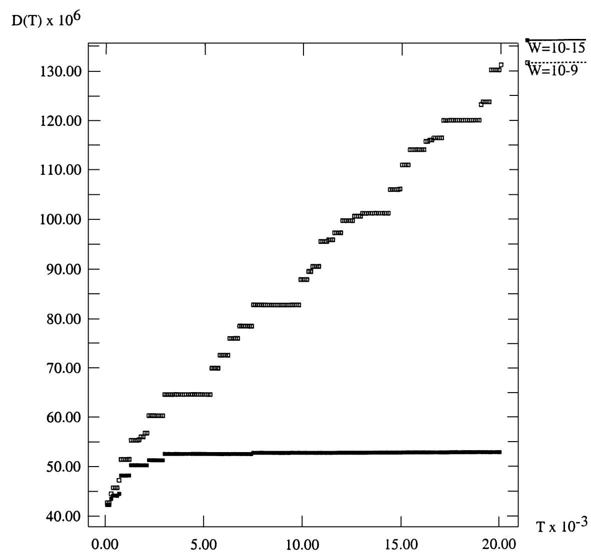

Fig. 1. Variation of the amplitude over time, calculated for local error tolerances $W = 10^{-9}$(upper), 10^{-15}(lower). $D(T) \equiv \Delta A_\Gamma(T)$, $\Gamma = [5, 0]$ (same trajectory as in Table 3).

3. The largest discrepancies between $\Delta A_\gamma(T)$ and $\Delta A_\gamma(4T)$ are observed for the mode a_Γ. Such discrepancies may be due to some kind of numerical drift of the amplitudes, seen for example in Fig. 1. The most dramatic discrepancy we observed is shown in Table 3. It was only observed for $W = 10^{-9}$ and was suppressed with $W = 10^{-15}$. Note, that for the times we have integrated such drifts are rare, moreover even in the worst cases they do no alter our quantitative results significantly.

From the above comparisons, we may conclude that the results obtained for $T = 250$ and with $W = 10^{-9}$ give reliable information on the change of amplitude of the various modes. Our observations that $\Delta A_\gamma(T)$ is attained early, and that $\Delta A_\gamma(4T)$ is only slightly larger than $\Delta A_\gamma(T)$ suggest non-decay of the mode a_Γ for longer times.

Table 3
$\Gamma = [5, 0]$. W is local error tolerance.

γ	$W = 10^{-9}$		$W = 10^{-15}$	
	$\Delta A_\gamma(T)$[a]	$\Delta A_\gamma(4T)$	$\Delta A_\gamma(T)$	$\Delta A_\gamma(4T)$
[5, 0]	6.4653×10^{-5}	1.3126×10^{-4}	5.2595×10^{-5}	5.2875×10^{-5}
[2, 0]	8.1811×10^{-3}	8.1830×10^{-3}	8.1813×10^{-3}	8.1836×10^{-3}
[8, 0]	3.6994×10^{-3}	3.7007×10^{-3}	3.6995×10^{-3}	3.7009×10^{-3}

[a]Time $T = 250$.

Also, the approximate proportionality of the ΔA_γ to ϵ and the results of the second set of experiments suggest that the ΔA_γ we corresponding to different integration times and W should be close for other values of ϵ as well.

4. Normal forms

We now propose an explanation for the stability of the near-monochromatic solutions. We employ formal and heuristic arguments, some of which form the basis of the rigorous statements of the next section. The theory is based on a Birkhoff normal form calculation. In particular, we bring the system to a Birkhoff normal form, which truncated to $O(\epsilon^3)$, has families of near-monochromatic solutions and several constants of motion that give information on the stability of such solutions.

The starting point is the Hamiltonian $H = H_0 + \epsilon H_1$ of (6) and (7). Since we are interested in explaining the numerical simulations, we consider Galerkin truncations to modes with $l \leq l_{\max} = 8$, i.e. the phase space is $\mathbb{R}^{2N}$, $N = 80$. At the end of this section we will show that the formal results we will obtain are valid for truncations that include higher order non-linearity and more spherical harmonic modes, and we also discuss the physical validity of the model studied numerically.

We will use the following properties of ω_γ and $I_{\gamma_1,\gamma_2,\gamma_3}$:

1. Let $\gamma = [l, m]$, and ω_γ the dispersion relation of (8). Then ω_γ satisfies
$$\omega_\gamma = \omega(l) \quad \text{and} \quad \omega(l) > 0, \quad \omega'(l) < 0, \quad \omega''(l) < 0; \quad \forall\, l > 0. \tag{11}$$
2. Let $\gamma = [l_i, m_i]$, $i = 1, 2, 3$. The coefficients $I_{\gamma_1,\gamma_2,\gamma_3}$ of the cubic Hamiltonian in (7) satisfy
$$I_{\gamma_1,\gamma_2,\gamma_3} \neq 0 \Rightarrow |l_1 - l_2| \leq l_3 \leq l_1 + l_2 \quad \text{and} \quad m_1 + m_2 + m_3 = 0, \quad \forall\, l_i \in \mathbb{Z}^+, \quad m_i = -l_i, \ldots, l_i, \quad i = 1, 2, 3. \tag{12}$$
3. Let ω_γ be as in (8). Consider the equation
$$\omega_{\gamma_1} + \omega_{\gamma_2} = \omega_{\gamma_3} + \omega_{\gamma_4}, \tag{13}$$
for $l_1, \ldots, l_4$ that satisfy
$$\exists l \in \mathbb{Z}^+ \quad \text{s.t.} \quad |l_1 - l_2| \leq l \leq l_1 + l_2 \quad \text{and} \quad |l_3 - l_4| \leq l \leq l_3 + l_4. \tag{14}$$
Then the only positive integers $l_1, \ldots, l_4 \leq l_{\max} = 8$ that satisfy (13) and (14) are the trivial solutions $l_1 = k_+, l_2 = k_-, l_3 = k_\pm, l_4 = k_\mp$.

Properties (1) and (2) have been verified for the particular problem in [1], but they are of wider applicability for waves on the sphere. The fact that the dispersion depends on l alone, and property (2) are consequences of spherical symmetry. The other properties in (1) make ω_γ analogous to ‘non-decay’ dispersion relations encountered, for instance, in plasma physics. Property (3) is verified numerically for the particular b, h we considered. We expect that if we fix any $l_{\max}$ and vary $\beta = h/b > 0$, the property will hold for an open (non-empty) set of β. However we can not estimate how far we are from solving the Diophantine equation (13). Note that such an estimate would not be uniform in $l_{\max}$. As $l \to \infty$, $\omega(l) \to \sqrt{l}$ and (13) and (14) has non-trivial solutions.

The next propositions are consequences of the above properties, and follow immediately from results of a normal form calculation found in [1], where we refer for the proofs. Here we add some remarks and corollaries that relate to the numerical experiments.

Proposition 1. *Consider the Galerkin truncations to modes with* $l \leq l_{\max} = 8$ *of the Hamiltonian system* (9) *with* $H = H_0 + \epsilon H_1$, H_0, H_1 *as in* (6) *and* (7). *Then properties* (1) *and* (2) *above hold and imply that there exists a formal canonical transformation* $\Phi^{(\epsilon S_1)} : \mathbb{R}^{2N} \to \mathbb{R}^{2N} (N = 80)$ *so that* $H \circ \Phi^{(\epsilon S_1)} = H_0 + \epsilon^2 G_2 + \mathrm{O}(\epsilon^3)$, *and* G_2 *is quartic in the canonical variables* (*terms* $\mathrm{O}(j)$ *are of order* $j + 2$).

The canonical transform is produced by the so called Lie-series method (see e.g. [16]), in particular $H \circ \Phi^{(\epsilon S_1)} = \exp Ad_{\epsilon S_1} H$, where S_1 is an appropriate function on $\mathbb{R}^{2N}$. The transform $\Phi^{(\epsilon S_1)}$ is a formal power series, and its domain will be specified in the next section. The gist of Proposition 1 is that properties (1) and (2) imply that there are no cubic resonances. Moreover we can bound away cubic resonances uniformly in $l_{\max}$ and therefore the proposition holds for arbitrary Galerkin truncations and for the infinite dimensional system as well.

Proposition 2. *Consider the Hamiltonian* $H \circ \Phi^{(\epsilon S_1)}$ *of Proposition* 1, *and the corresponding Hamiltonian system in the Galerkin truncation to modes with* $l \leq l_{\max} = 8$. *Then property* (3) *holds and implies that there exists a formal canonical transformation* $\Phi^{(\epsilon^2 S_2)} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ *such that* $H \circ \Phi^{(\epsilon S_1)} \circ \Phi^{(\epsilon^2 S_2)} = H_0 + \epsilon^2 N_2 + \mathrm{O}(\epsilon^3)$, *and the (resonant quartic) term* N_2 *is a linear combination of monomials* $a_{\gamma_1} a_{\gamma_2} a^*_{\gamma_3} a^*_{\gamma_4}$ *with* $l_3 = l_1$ *and* $l_4 = l_2$(*or* $l_3 = l_1, l_4 = l_2$), *and* $m_1 + m_2 = m_3 + m_4$.

The canonical transform is again produced using the Lie-series and $H \circ \Phi^{(\epsilon S_1)} \circ \Phi^{(\epsilon^2 S_2)} = \exp Ad_{\epsilon^2 S_2} H \circ \Phi^{(\epsilon S_1)}$, with an appropriate function S_2. Proposition 2 holds for finite dimensional systems, because of the inapplicability of property (3) in the limit $l_{\max} \to \infty$.

We now consider properties of the Hamiltonian system (9) with the quartic normal form Hamiltonian $\overline{H} = H_0 + \epsilon^2 N_2$ in the phase space of the modes with $l \leq l_{\max} = 8$. We introduce the action variables $J_\gamma = a_\gamma a^*_\gamma$ and the angles $\theta_\gamma = Arg(a_\gamma), \forall \gamma$ with $l \leq l_{\max} = 8$. Away from the hyper-planes $a_\gamma = 0$ the Hamiltonian $\bar{H}$ is real analytic in the J_γ, θ_γ. Hamilton's equations become

$$\dot{J}_\gamma = -\frac{\partial \bar{H}}{\partial \theta_\gamma}, \quad \dot{\theta}_\gamma = \frac{\partial \bar{H}}{\partial J_\gamma}. \tag{15}$$

From Proposition 2 we immediately have the following.

Corollary 1. *The Hamiltonian* $\bar{H} = H_0 + \epsilon^2 N_2$, *restricted to the subspace of axisymmetric solutions is independent of the angles.*

Proof. By Proposition 2 the Hamiltonian $\bar{H}$, restricted to the axisymmetric solutions, consists of monomials of the type $a_{[l_1,0]} a_{[l_2,0]} a^*_{[l_1,0]} a^*_{[l_2,0]} = J_{[l_1,0]} J_{[l_2,0]}$. □

Corollary 2. *The Hamiltonian system corresponding to* $\overline{H} = H_0 + \epsilon^2 N_2$ *has* $l_{\max}$ *constants of motion* I_l *given by*

$$I_l = \frac{1}{2l+1} \sum_{m=-l}^{l} J_{[l,m]}, \quad l = 1, 2, \ldots, l_{\max}.$$

Proof. We make a canonical transformation to a new set of variables $\phi_{[l,j]}$ and $I_{[l,j]}$, $l = 1, \ldots, l_{\max}$, $j =$

$1, \dots, 2l+1$ defined by

$$\phi_{[l,1]} = \sum_{m=-l}^{l} \theta_{[l,m]}, \quad l = 1, \dots, l_{\max},$$

$$\phi_{[l,j]} = \sum_{m=-l}^{l} k^m_{[l,j]} \theta_{[l,m]}, \quad l = 1, \dots, l_{\max}, \quad j = 2, 3, \dots, 2l+1,$$

where the $k^i_{[l,j]}$ are chosen so that $\phi_{[l,1]}$ is perpendicular to all the $\phi_{[l,j]}$, $j = 2, 3, \dots, 2l+1$ and the $\phi_{[l,j]}$, $j = 2, 3, \dots, 2l+1$ are orthogonal to each other. Also,

$$I_{[l,1]} = \frac{1}{2l+1} \sum_{m=-l}^{l} J_{[l,m]}, \quad l = 1, \dots, l_{\max},$$

$$I_{[l,j]} = \frac{1}{\sum_{m=-l}^{l} (k^m_{[l,j]})^2} \sum_{m=-l}^{l} k^m_{[l,j]} J_{[l,m]}, \quad l = 1, \dots, l_{\max}, \quad j = 2, 3, \dots, 2l+1.$$

From (15), Hamilton's equations in the new variables are

$$\dot{I}_{[l,j]} = -\frac{\partial \bar{H}}{\partial \phi_{[l,j]}}, \quad \dot{\phi}_{[l,j]} = \frac{\partial \bar{H}}{\partial I_{[l,j]}}, \tag{16}$$

for $l = 1, \dots, l_{\max}$, $j = 1, \dots, 2l+1$. By Corollary 2 we see that N_2 does not depend on the angles $\phi_{[l,1]}$ and therefore by (15) the $I_{[l,1]}$ are constant. Setting $I_l = I_{[l,1]}, l = 1, \dots, l_{\max}$ we have the proposition. □

For axisymmetric solutions, Corollary 1 implies that the amplitude of the modes should change slowly, to $O(\epsilon)$ over time of $O(\epsilon^{-2})$, so that solutions starting from near-monochromatic initial conditions should stay near-monochromatic for long time. For general solutions, Corollary 2 implies that a near-monochromatic solution cannot decay too fast to modes with different l wave-numbers. However the corollary does not give us information on the transfer of energy to modes with the same l wave-number as that of the initial condition. In some cases we can account for the observed behavior by using a heuristic argument that is based on the following statement we have shown in [1].

Proposition 3. *Let* $\Gamma = [l_\Gamma, m_\Gamma]$ *be any mode index with* $l_\Gamma \le l_{\max} = 8$. *Then the plane* $V_\Gamma = \{a \in \mathbb{C}^N : a_\gamma = 0, \quad \gamma \ne \Gamma\}$ *is invariant under the flow of the system corresponding to* $\bar{H}$.

The idea is now to show that under the flow of the Hamiltonian $\bar{H} = H_0 + \epsilon^2 N_2$ any linear instabilities of the orbits of Proposition 3 are $O(\epsilon^3)$. We consider Hamilton's equations for $\bar{H}$. For a given Γ (i.e. $a_\Gamma(0) = O(1)$) we substitute $a_\Gamma(t) = A_\Gamma e^{-i\omega_\Gamma t}$ and keep in the equation only terms that are linear in all the a_γ with $\gamma \ne \Gamma$. The resulting system would be the variational equation around the periodic orbit $a_\Gamma(t) = A_\Gamma e^{-i\omega_\Gamma t}$, $a_\gamma(t) \equiv 0$ for $\gamma \ne \Gamma$, off-course $a_\Gamma(t)$ is only $O(\epsilon^3)$ close to what we prescribe. On the invariant plane V_Γ the system is integrable. Near the origin it looks like an oscillator with amplitude dependent frequency. We thus look for possible parametric instability along the other directions. The equations for $\dot{a}_\gamma, \gamma \ne \Gamma$, form a non-autonomous linear system with periodic coefficients, of period ω_Γ. For each $\gamma_i \ne \Gamma$, $\dot{a}_{\gamma_i}$ is proportional to terms of type (a) $a_{\gamma_j} a_\Gamma a^*_\Gamma$, and (b) $a^*_{\gamma_j} a_\Gamma a_\Gamma$. We can eliminate non-resonant terms in the equations for $\dot{a}_\gamma$ by averaging, i.e. by writing the equations for the amplitudes $A_\gamma(t)$ defined by $a_\gamma(t) = A_\gamma(t) e^{-i\omega_\gamma t}$ and time-averaging. The terms of type (a) that remain for each $\dot{A}_{\gamma_i}(t)$ are those with $\omega_{\gamma_j} = \omega_{\gamma_i}$ and $m_{\gamma_j} = m_{\gamma_i}$. Such terms only alter the frequencies of $a_\gamma(t)$ and cannot give rise to hyperbolic directions. Resonant terms of type (b) that remain after the averaging must satisfy

$\omega_{\gamma_i} + \omega_{\gamma_j} = 2\omega_\Gamma$ and $m_{\gamma_i} + m_{\gamma_j} = 2m_\Gamma$. By property (3) these conditions require $\omega_{\gamma_i} = \omega_{\gamma_j} = \omega_\Gamma$. Therefore instability can occur only along directions with $l = l_\Gamma$. In addition, if $m_\Gamma = l$ or $-l$, then no $\mathrm{O}(\epsilon^2)$ terms can induce parametric instability.

The formal results obtained so far also hold for the Hamiltonian flow of the quartic Hamiltonian $H_q = H_0 + \epsilon H_1 + \epsilon^2 H_2$ (with $l_{\max} = 8$). Comparing the normal forms obtained from the cubic and quartic Hamiltonians also clarifies the dependence of our results on the depth parameter β. First, using H_q, Proposition 1 still holds, with the quartic part $\epsilon^2 G_2$ replaced by $\epsilon^2 G_{2,q} = \epsilon^2 G_2 + \epsilon^2 H_2$. The function S_1 is computed from H_1 and remains the same. Proposition 2 also holds, with $\epsilon^2 N_2$ replaced by $\epsilon^2 N_{2,q} = \epsilon^2 N_2 + \epsilon^2 \overline{H}_2$, where $\bar{H}_2$ is the normal form part of H_2 Note that $\epsilon^2 N_{2,q}$ is computed in [1], and like N_2 it is a linear combination of monomials $a_{\gamma_1} a^*_{\gamma_2} a_{\gamma_3} a^*_{\gamma_4}$ with $l_3 = l_1$ and $l_4 = l_2$ (or $l_3 = l_1, l_4 = l_2$), and $m_1 + m_2 = m_3 + m_4$. From this observation it easily follows that Corollaries 1 and 2 and Proposition 3 also hold for the quartic Hamiltonian H_q. Thus our formal results hold for the quartic Hamiltonian, and also for higher order non-linearities since no quintic or higher order terms play any role in the arguments.

To make the dependence of the normal form on β explicit we define $\tilde{H}_1$ and $\tilde{H}_2$ by $H_1 = \beta \tilde{H}_1$ and $H_2 = \beta^2 \tilde{H}_2$ respectively (as we have already noted such a factorization is possible by the dimensional analysis of [1]). Starting with $H_q = H_0 + \epsilon\beta\tilde{H}_1 + \epsilon^2\beta^2\tilde{H}_2$ we seek a function $\tilde{S}_1$ so that $\exp Ad_{\epsilon\tilde{S}_1} H_q$ does not contain cubic terms. Using the definition of $\exp Ad_{\epsilon\tilde{S}_1} H_q$ we must then require that $\tilde{S}_1$ satisfy the 'cohomology equation' $[\epsilon\tilde{S}_1, H_0] + \epsilon\beta\tilde{H}_1 = 0$. On the other hand, S_1 is also chosen to eliminate cubic terms in $\exp Ad_{\epsilon S_1} H_q$ and satisfies the same equation, thus $\tilde{S}_1 = S_1$. Since there are no cubic resonances $\tilde{S}_1$ can be found and the transformed Hamiltonian simplifies to $\exp Ad_{\epsilon\tilde{S}_1} H_q = H_0 + (1/2)\epsilon^2\beta[\tilde{H}_1, \tilde{S}_1] + \epsilon^2\beta^2\tilde{H}_2 + \mathrm{O}(\epsilon^3)$, and since $\tilde{S}_1 = S_1$ we have that $\epsilon^2 G_{2,q} = (1/2)\epsilon^2\beta[\tilde{H}_1, S_1] + \epsilon^2\beta^2\tilde{H}_2$. From the cohomology equation defining $\tilde{S}_1$ above we have that, dimensionally, $\tilde{S}_1 \sim (\beta/\omega)\tilde{H}_1$, where ω is a typical frequency. Using the dispersion relation in (8) we have that for β small, $\omega_\gamma^2 = l(l+1)\beta + \mathrm{O}(\beta^2)$. Hence, dimensionally, $\tilde{S}_1 \sim \sqrt{\beta}\tilde{H}_1$, and the dominant part of $\epsilon^2 G_{2,q}$ is the one arising from H_1 for β sufficiently small.

Since the arguments of this section are based on the properties of the quartic resonant terms, we cannot consider β arbitrarily small. Note that the dimensional expression $\tilde{S}_1 \sim (\beta/\omega)\tilde{H}_1$ gives meaningful information only in the case where we are sufficiently far from the cubic resonances. A more precise estimate for the size of $\tilde{S}_1$ is $\tilde{S}_1 \sim |\omega_{\gamma_1} - \omega_{\gamma_2} - \omega_{\gamma_3}|^{-1}\beta\tilde{H}_1$, with $\gamma_1, \gamma_2, \gamma_3$ the modes that are closer to the resonance. As $\beta \to 0$ the rate of approach to resonance is faster than β, and we cannot define a function $\tilde{S}_1$ eliminating all cubic terms (e.g. for $\beta = 0.2$ we have $|\omega_{\gamma_1} - \omega_{\gamma_2} - \omega_{\gamma_3}|^{-1} = 0.1$ for the modes $\gamma_1, \gamma_2, \gamma_3$ closest to resonance). Thus as β decreases the cubic Hamiltonian becomes a better physical model, for the dimensional reasons we already discussed, but also, more fundamentally, due to the approach to cubic resonances. In a range near $\beta = 0.2$, the dimensional analysis suggests that the cubic Hamiltonian gives a reasonable approximation, while at the same time we are not too close to the cubic resonances.

Remark 1. The theory of this section does not rule out the possibility of decay in time scales lower than $\mathrm{O}(\epsilon^{-2})$ of a non-axisymmetric near-monochromatic solution with $a_{[l_\Gamma, m_\Gamma]} = \mathrm{O}(1)$ to modes with $l = l_\Gamma$ (except in the case where $m_\Gamma = \pm l_\Gamma$). Thus we cannot rule out the possibility that the quartic terms of H_2 cause such a decay, altering the numerical results in a qualitative way. We hope to clarify this issue using some recent results in which we give sufficient conditions on the quartic normal form for such decay not to occur. Verifying these conditions for the quartic Hamiltonian would complete the theoretical explanation of the numerical observations and also show that the stability of the monochromatic orbits is not affected by the addition of the quartic terms.

To see the dependence of the formal results on the size of the Galerkin truncation we note that properties (1) and (2) hold for arbitrary $l_{\max}$, while condition (3) must be checked numerically for each choice of $l_{\max} > 8$. Thus, subject to verification of (3), all the results of this section will hold for larger Galerkin truncations.

5. Stability estimates

The normal form argument of the previous section gives us a partial explanation of the behavior we observed in the numerical experiments. In this chapter we obtain bounds for the variation of (the square of) the mode amplitudes and estimate the time and the size of initial conditions for which such bounds are valid. With these results the normal form argument of the previous section becomes quantitative and rigorous. We summarize the results in two theorems, for axisymmetric and general solutions respectively.

Since all functions in the problem are given explicitly as formal power series on $\mathbb{R}^{2N}$, it is natural to work with their extension to power series on $\mathbb{C}^{2N}$. Specifically, consider $\mathbb{C}^{2N}$ with coordinates $z = (z_1, z_2, \dots, z_{2N})$. Let $\varrho = (\rho_1, \dots, \rho_{2N})$ and $k = (k_1, \dots, k_{2N})$ be multi-indices. We define the polydiscs D_ϱ by

$$D_\varrho = \{z \in \mathbb{C}^{2N} : \quad |z_i| \le \rho_i\},$$

and we consider formal power series on D_ϱ, i.e. expressions $\sum_{k\ge 0} f_k z^k$, $f_k \in \mathbb{C}$ (using multi-index notation, and also $k \ge 0$ to denote multi-indices for which $k_i \ge 0$, $i = 1, \dots, 2N$). Let $\|\sum_{k\ge 0} f_k z^k\|_\varrho = \sum_k |f_k| \varrho^k$ and define the set $\mathcal{A}(D_\varrho)$ of analytic functions on D_ϱ by

$$\mathcal{A}(D_\varrho) = \left\{ \sum_{k\ge 0} f_k z^k : \quad \left\| \sum_{k\ge 0} f_k z^k \right\|_\varrho < \infty \right\}.$$

The spaces $\mathcal{A}(D_\varrho)$, with the norm $\| \ \|_\varrho$ are Banach algebras (see e.g. [17] for their basic properties). For simplicity, here we will consider special polydiscs D_ϱ in which $\varrho_i = \rho, \forall i = 1, \dots, 2N$. We denote these polydiscs by D_ρ. The corresponding spaces of analytic functions and norms are denoted by $\mathcal{A}(D_\rho)$ and $\| \ \|_\rho$ respectively. In the Appendix we state three lemmas that follow directly from the above definitions, and that are used in the Proofs of Theorems 1 and 2 below.

The notation of Theorem 1 on axisymmetric solutions is as follows. The phase space is $\mathbb{R}^{2N}$, $N = l_{\max} = 8$, and the coordinates x_i, q_i, p_i are defined by $x_i = q_i = q_{[i,0]}$, $x_{i+N} = p_i = p_{[i,0]}$, $i = 1, \dots, l_{\max}$ (see Section 2 for the $q_{[i,0]}$, $p_{[i,0]}$). Also, $x = [x_1, \dots, x_{2N}]$. The Hamiltonian will be denoted by h and $h = h_0 + \epsilon h_1$, with h_0, h_1 the restriction of H_0, H_1 respectively to the subspace of axisymmetric solutions V_A. Note that the canonical transformations of the Propositions 1 and 2 leave V_A invariant. We denote the restrictions of S_1, S_2 to V_A by s_1 and s_2 respectively (i.e. the notation is that of Sections 2 and 4, with functions restricted to V_A denoted by small letters). The theorem will involve the constants of Lemmas A.1–A.3 of the Appendix and also c_1 and c_2, the bounds from the cubic and quartic resonances respectively (for $l \le l_{\max} = 8$). Specifically, c_1, satisfies $|\omega(l_1) - \omega(l_2) - \omega(l_3)|^{-1} \le c_1, \forall l_1, l_2, l_3 \in \mathcal{J}_{l_{\max}} = \{1, 2, \dots, l_{\max}\}$, while c_2 satisfies $|\omega(l_1) - \omega(l_2) - \omega(l_3) - \omega(l_4)|^{-1} \le c_2, \forall l_1, l_2, l_3, l_4 \in \mathcal{J}_{l_{\max}}$, and $|\omega(l_1) + \omega(l_2) - \omega(l_3) - \omega(l_4)|^{-1} \le c_2$, for all quartets (l_1, l_2, l_3, l_4), $l_1, l_2, l_3, l_4 \in \mathcal{J}_{l_{\max}}$, that are not of the form (l_1, l_2, l_1, l_2) or (l_1, l_2, l_2, l_1).

Theorem 1. *Consider axisymmetric solutions of the equations* (9) *with the notation above. Let* $1 \ge \rho > 5\delta > 0$ *and consider initial conditions* $x(0) \in D_{\rho_1}$, *with* $\rho_1 \le (\sqrt{2} - 1)(\rho - 5\delta)$. *Also assume that* $\epsilon > 0$ *is such that*

$$\lambda_1 = \epsilon c_1 \frac{e^2 c^2}{\delta^2} \|h_1\|_\rho < 1, \quad \lambda_2 = (\lambda_1)^2 \frac{c_2}{c_1 e^2} < 1, \quad \epsilon\mu \le \frac{2 - \sqrt{2}}{2}(\rho - 5\delta), \quad \frac{e^2 \delta}{2\kappa_1 \|r\|_{\rho - 2\delta}} \le 1, \tag{17}$$

with $\mu = \mathrm{O}(1)$, $r = \mathrm{O}(\epsilon^3)$ *defined in Lemmas* 1 *and* 2 *below. Then, properties* (1)–(3) *of Section* 4 *imply that the actions* $J_i = q_i^2 + p_i^2$, $i = 1, \dots, N$ *satisfy*

$$|J_i(t) - J_i(0)| \le 8\epsilon\mu + 4\kappa_1 \frac{\|r\|_{\rho - 2\delta}}{\epsilon^2} = \mathrm{O}(\epsilon), \quad \forall t \in [0, \delta\epsilon^{-2}). \tag{18}$$

In Theorem 2 for general solutions of (9) below, we consider modes up to $l_{\max} = 8$. The phase space will be $\mathbb{R}^{2N}$ with $N = 80$ and we use the coordinates $y_{l^2+l+m} = q_{[l,m]}$, $y_{N+l^2+l+m} = p_{[l,m]}$ (as in Section 2). The notation for the Hamiltonian and the functions generating the canonical transformations are as in Sections 2 and 4. The constants c_1, c_2 are as in the axisymmetric case.

Theorem 2. *Consider solutions of the equations* (9) *with the notation above. Let* $1 \geq \rho > 5\delta > 0$ *and consider initial conditions* $y(0) \in D_{\rho_2}$, *with* $\rho_2 \leq 3^{-1}(4l_{\max}+2)^{-3/2}(\rho - 5\delta)$. *Moreover, let* ϵ *be such that*

$$\Lambda_1 = \epsilon c_1 \frac{e^2 c^2}{\delta^2}\|H_1\|_\rho < 1, \quad \Lambda_2 = (\Lambda_1)^2 \frac{c_2}{c_1 e^2} < 1, \quad \epsilon M \leq 3^{-1}(4l_{\max}+2)^{-32}(\rho - 5\delta),$$
$$\frac{e^2\delta}{\sqrt{2}(2l_{\max}+1)\kappa_1\|R\|_{\rho-2\delta}} \leq 1, \tag{19}$$

with $M = \mathrm{O}(1)$, $R = \mathrm{O}(\epsilon^3)$ *defined below. Then, properties* (1)–(3) *of Section* 4 *imply that the variables* $I_i = (2l+1)^{-1}\sum_{m=-l}^{l}(q_{[l,m]}^2 + p_{[l,m]}^2)$, $i = 1, \ldots, N$ *satisfy*

$$|I_i(t) - I_i(0)| \leq 4\epsilon M + \frac{\kappa_1\|R\|_{\rho-2\delta}}{2\delta\epsilon^2} = \mathrm{O}(\epsilon), \quad \forall\, t \in [0, \delta\epsilon^{-2}). \tag{20}$$

Remark 2. *The Hamiltonians of the finite dimensional Galerkin systems are positive definite near the origin, and therefore, for sufficiently small initial conditions, solutions exist for all time. This fact is not used in the theorems. Rather, both theorems only require local existence of solutions, and the conclusion of the theorems can be considered as a theorem of existence for time of* $\mathrm{O}(\epsilon^{-2})$.

Remark 3. *In the case of axisymmetric solutions, the result of* [8] *implies that, if* N_2 *is definite as a quadratic from in the actions* J_γ, *the properties* (1)–(3) *of Section* 4 *guarantee stability of near-monochromatic solutions for exponentially long time.*

Remark 4. *Comparing the definitions of the* c_1, c_2 *with the properties* (1)–(3) *of Section* 4, *we see that* $(c_1)^{-1}$ *and* $(c_2)^{-1}$ *are the bounds from the resonances (cubic and quartic respectively) the were absent. For* $l_{\max} = 8$, *we found numerically that* c_1, c_2 *can be chosen as* $c_1 = 10$ *and* $c_2 = 143$.

The proofs of the two theorems are very similar. The machinery is quite standard, and has been developed in the context of Nekhoroshev-type results (see e.g. [18]), averaging, KAM theorems etc. One distinct feature of the present proof is that the canonical transformations are around the origin (not around an invariant torus) and some care is needed in dealing with the singularity of the change to action-angle variables at the hyperplanes $a_\gamma = 0$.

Proof of Theorem 1. The proof is broken into four parts. In Lemma 1 we estimate the domain of the canonical transformation and the remainder of the quartic normal form. Lemma 2 gives us the distance between the original and the transformed variables. In Lemma 3 we find times for which the trajectories of the equations stay within the domain of the transformation. In Lemma 4 we use the normal form equations to estimate the drift of the actions for initial conditions and times suggested by the previous lemmas. □

Lemma 1. *Let* $\rho > 5\delta > 0$ *and* $\lambda_1, \lambda_2 < 1$ *as in* (17). *Then properties* (1)–(3) *of Section* 4 *imply that there is a canonical transformation* $\Phi^{(\epsilon s_1, \epsilon^2 s_2)} : \mathcal{A}(D_{\rho-2\delta}) \to \mathcal{A}(D_\rho)$ *so that* $h \circ \Phi^{(\epsilon s_1, \epsilon^2 s_2)} = h_0 + \epsilon^2 n_2 + r$, *with* n_2 *the quartic part of the normal form Hamiltonian of Proposition* 2 *(restricted to the space axisymmetric solutions* V_A*), and* r *the remainder i.e. the terms of* $\mathrm{O}(\epsilon^3)$. *The remainder satisfies*

$$\|r\|_{\rho-2\delta} \leq (\lambda_1)^3\alpha, \tag{21}$$

with α *given in the proof.*

Proof. Let s_i, $i = 1, 2$ be the functions that generate the canonical transformations on V_A, and let $L_i f = Ad_{\epsilon^i s_i} f$, for any function f. With this notation we have

$$h \circ \Phi^{(\epsilon s_1)} = \exp Ad_{\epsilon s_1} h = h_0 + \epsilon^2 g_2 + r_1^0 + r_1^1,$$

with

$$g_2 = \frac{1}{2} L_1 h_1, \quad r_1^0 = \sum_{k=3}^{\infty} \epsilon^k \frac{L_1^k}{k!} h_0, \quad r_1^1 = \sum_{k=2}^{\infty} \epsilon^k \frac{L_1^k}{k!} \epsilon h_1.$$

Also,

$$h \circ \Phi^{(\epsilon s_1, \epsilon^2 s_2)} = \exp Ad_{\epsilon^2 s_2} \exp Ad_{\epsilon s_1} h = h_0 + \epsilon^2 n_2 + r,$$

with

$$n_2 = g_2 + L_2 h_0, \quad r = r_1^0 + r_1^1 + r_2^0 + r_2^1, \quad r_2^0 = \epsilon^2 L_2(\epsilon^2 g_2 + r_1^0 + r_1^1), \quad r_2^1 = \sum_{k=3}^{\infty} \epsilon^{2k} \frac{L_2^k}{k!} \exp Ad_{\epsilon s_1} h.$$

We use Lemmas A.2 and A.3 of the Appendix and $\|s_1\|_\rho \le c_1 \|h_1\|_\rho$, $\|s_2\|_\rho \le c_2 \|h_2\|_\rho$ to bound the different terms of r in $\mathcal{A}(D_{\rho-2\delta})$. For example, we have

$$\|g_2\|_{\rho-2\delta} \le \frac{\tilde{c} c_1}{2\delta^2} \|h_1\|_\rho^2, \quad \|r_1^0\|_{\rho-\delta} < \frac{(\lambda_1)^3}{1-\lambda_1} \|h_0\|_\rho, \quad \text{and} \quad \|r_1^1\|_{\rho-\delta} < \frac{(\lambda_1)^3}{1-\lambda_1} \frac{\delta^2}{\tilde{c} c_1}, \quad \text{both for} \quad \lambda_1 < 1.$$

The condition $\lambda_1 < 1$ guarantees the convergence of the first canonical transformation. Similarly we estimate n_2, while for r_2^0 we use

$$\|L_2 g_2\|_{\rho-2\delta} \le \frac{2\tilde{c}}{\delta^2} \|s_2\|_{\rho-\delta} \|g_2\|_{\rho-\delta} \le \frac{\tilde{c} c_2}{\delta^2} \|L_1 h_1\|_{\rho-\delta}^2 \le \frac{\tilde{c} c_2}{\delta^2} \left(\frac{\epsilon \tilde{c}}{\delta^2} \|s_1\|_{\rho-\delta 2} \|h_1\|_\rho \right)^2 \le \frac{\tilde{c} c_2}{\delta^2} \left(\frac{2\tilde{c} c_1}{\delta^2} \|h_1\|_\rho^2 \right)^2,$$

and likewise for the rest of r_2^0, using the bounds for r_1^0, r_1^1. Collecting the calculations we have

$$\|r_2^0\|_{\rho-2\delta} \le (\lambda_1)^4 \frac{4 c_2 \delta^2}{\tilde{c} c_1^2 e^8} + (\lambda_1)^2 \frac{2 c_2}{c_1 e^4} \left(\frac{(\lambda_1)^3}{1-\lambda_1} \|h_0\|_\rho + \frac{(\lambda_1)^3}{1-\lambda_1} \frac{\delta^2}{\tilde{c} c_1} \right)$$

and also, using Lemma A.3 of the Appendix,

$$\|r_2^1\|_{\rho-2\delta} \le \frac{(\lambda_2)^2}{(1-\lambda_1)(1-\lambda_2)} \|h\|_\rho, \quad \text{for} \quad \lambda_2 < 1.$$

The condition $\lambda_2 < 1$ is needed for the convergence the second canonical transformation. Collecting all the terms of the remainder we obtain (21) with

$$\alpha = \left(10 + (\lambda_1)^2 \frac{c_2}{2c_1} \right) \left(\|h\|_\rho + \frac{1}{8Nc_1} \right) + \frac{100\lambda_1}{e^4} \left(\frac{c_2}{c_1} \right)^2 \|h\|_\rho + \frac{\lambda_1 c_2}{2c_1^2 N e^2}. \qquad \square$$

Lemma 2. *Let* $\rho > 5\delta > 0, \lambda_1, \lambda_2 < 1$ *as in Lemma* 1 *and consider the restriction* $\Phi^{(\epsilon s_1, \epsilon^2 s_2)} : \mathcal{A}(D_{\rho-4\delta}) \to \mathcal{A}(D_{\rho-2\delta})$. *Then*

$$|x_i - x_i \circ \Phi^{(\epsilon S_1, \epsilon^2 S_2)}| < \epsilon\mu, \quad \forall\, i = 1, \ldots, 2N, \tag{22}$$

with μ *defined below.*

Proof. The first canonical transformation $\Phi^{(\epsilon s_1)}$ is the time-1 map of the Hamilton's equations for ϵs_1, i.e. $\mathrm{d}q_i/\mathrm{d}\tau = -\epsilon\partial s_1/\partial p_i$, $\mathrm{d}p_i/\mathrm{d}\tau = \epsilon\partial s_1/\partial q_i$. Suppose $x \in D_{\rho-4\delta}$ at $\tau = 0$. Then the smallest time τ_m for the trajectories of Hamilton's equations for ϵs_1 to be outside $D_{\rho-3\delta}$ is $\tau_m = \delta(\sup_{x\in D_{\rho-3\delta}, j=1,\dots,2N}|\partial S_1/\partial x_j|)^{-1}$. We easily see that $\lambda_1 < 1$ implies $1 < \tau_m$. Similarly, if we consider the time-1 map of Hamilton's equations for s_2 we have that if $\lambda_2 < 1$ and $x \in D_{\rho-3\delta}$ at $\tau = 0$ then $x \circ \Phi^{(\epsilon^2 s_2)} \in D_{\rho-2\delta}$. Thus, $\forall\, i = 1, \dots, 2N$ we have

$$|x_i - x_i \circ \Phi^{(\epsilon s_1, \epsilon^2 s_2)}| \le \epsilon \max_{j=1,\dots,2N} \left\| \frac{\partial s_1}{\partial x_j} \right\|_{\rho-3\delta} + \epsilon^2 \max_{j=1,\dots,2N} \left\| \frac{\partial s_2}{\partial x_j} \right\|_{\rho-2\delta}$$
$$\le \epsilon \left(\frac{c_1}{3\delta} \|h_1\|_\rho + \frac{\epsilon c_2}{\delta} \|g_2\|_{\rho-\delta} \right) = \epsilon\mu. \tag{23}$$

□

Lemma 3. *Let ρ, δ be as in Lemmas 1 and 2, and define $\tilde{x}_i = x_i \circ \Phi^{(\epsilon s_1, \epsilon^2 s_2)}$, $i = 1, \dots, 2N$ and $\tilde{x} = [\tilde{x}_1, \dots, \tilde{x}_{2N}]$. Assume that $\tilde{x}(0) \in D_{\tilde{\rho}/2}$, with $\tilde{\rho} \le 1/\sqrt{2}(\rho - 5\delta)$. Then*

$$\tilde{x}(t) \in D_{\tilde{\rho}+\delta\sqrt{2}}, \quad \forall\, t \in \left[0, \frac{\delta^2}{2\kappa_1 \|r\|_{\rho-2\delta}}\right).$$

Proof. Let $\tilde{q}_i = q_i \circ \Phi^{(\epsilon s_1, \epsilon^2 s_2)}$, $\tilde{p}_i = p_i \circ \Phi^{(\epsilon s_1, \epsilon^2 s_2)}$ and define $\tilde{J}_i = \tilde{q}_i^2 + \tilde{p}_i^2$, $\tilde{A}_i = \tilde{J}_i^{1/2}$, $i = 1, \dots, N$. From $\tilde{q}(0) \in D_{\tilde{\rho}/2}$ we have $\tilde{A}_i(0) < \tilde{\rho}/\sqrt{2}$. Then, if $\tilde{J}_i \ne 0$ we have

$$\frac{\mathrm{d}\tilde{A}_i}{\mathrm{d}t} = 2\tilde{A}_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \tilde{J}_i = 4\tilde{A}_i^{-1} \left(-\frac{\partial r}{\partial \tilde{q}_i} \tilde{p}_i + \frac{\partial r}{\partial \tilde{p}_i} \tilde{q}_i \right)$$

hence

$$\left| \frac{\mathrm{d}\tilde{A}_i}{\mathrm{d}t} \right| \le \frac{1}{4} \left(\left| \frac{\partial r}{\partial \tilde{q}_i} \right| + \left| \frac{\partial r}{\partial \tilde{p}_i} \right| \right). \tag{24}$$

Note that since $\tilde{q}_i(t)$, $\tilde{p}_i(t)$ are analytic in time (24) also holds for the times for which $\tilde{J}_i = 0$. Now, let T_0 be such that $\tilde{A}_i(t) \le \tilde{\rho}/\sqrt{2} + \delta/2$, $\forall\, t \in [0, T_0)$. Then for $t \in [0, T_0)$ we have

$$\left| \frac{\mathrm{d}\tilde{A}_i}{\mathrm{d}t} \right| \le \sup_{\tilde{A}_i < \frac{\sqrt{2}\tilde{\rho}+\delta}{2}} \left(\left| \frac{\partial r}{\partial \tilde{q}_i} \right| + \left| \frac{\partial r}{\partial \tilde{p}_i} \right| \right),$$

and using the assumption $\tilde{\rho} < 1/\sqrt{2}(\rho - 5\delta)$ we have

$$\left| \frac{\mathrm{d}\tilde{A}_i}{\mathrm{d}t} \right| \le \sup_{\tilde{x}\in D_{\rho-4\delta}} \left(\left| \frac{\partial r}{\partial \tilde{q}_i} \right| + \left| \frac{\partial r}{\partial \tilde{p}_i} \right| \right) \le \frac{\kappa_1}{\delta} \|r\|_{\rho-2\delta}. \tag{25}$$

Thus $T_0 > \delta^2 (2\kappa_1 \|r\|_{\rho-2\delta})^{-1}$ and the conclusion follows. □

Lemma 4. *Let ρ_1, $\epsilon\mu$, $\tilde{\rho}$ as in Lemmas 1 and 3, and assume that*

$$\epsilon\mu \le \frac{2-\sqrt{2}}{2}(\rho - 5\delta), \quad \textit{and} \quad \frac{\epsilon^2\delta}{2\kappa_1\|r\|_{\rho-2\delta}} \ge 1. \tag{26}$$

Then axisymmetric solutions of (9) with initial condition $x(0) \in D_{\rho_1}$ satisfy

$$|J_i(t) - J_i(0)| \le 8\epsilon\mu + 4\kappa_1 \frac{\|r\|_{\rho-2\delta}}{\epsilon^2}, \quad \forall\, t \in [0, \delta\epsilon^{-2}), \quad \forall\, i = 1, \dots, N. \tag{27}$$

Proof. By the assumptions on $\rho_1, \epsilon\mu$, we have $x(0) \in D_{\rho_1} \subseteq D_{\tilde{\rho}/2-\epsilon\mu}$ and therefore, by Lemma 2 and 3 applies and we have

$$\tilde{x}(t) \in D_{\tilde{\rho}+\delta/\sqrt{2}}, \quad \forall\, t \in [0, T_0), \quad \text{with} \quad T_0 = \frac{\delta^2}{2\kappa_1 \|r\|_{\rho-2\delta}}.$$

By the assumption on $\epsilon\mu$ in (26) and Lemma 2 we then have that $x(t) \in D_{\rho-4\delta}, \forall\, t \in [0, T_0)$. Thus for $x(0) \in D_{\rho_1}$ trajectories stay within the domain of the canonical transformation, for all times up to T_0. The second inequality of (25) implies that $\delta/\epsilon^2 < T_0$, so that using (24), the proof of Lemma 3, and the fact that $|q_i(t)|, |p_i(t)| < \rho \leq 1, \forall\, t \in [0, \delta/\epsilon^2)$ and $i = 1, \dots, N$, we have

$$\left| \frac{\mathrm{d}\tilde{J}_i}{\mathrm{d}t} \right| \leq \frac{\kappa_1}{\delta} \|r\|_{\rho-2\delta}, \quad \forall\, t \in [0, \delta\epsilon^{-2})$$

hence

$$|\tilde{J}_i(t) - \tilde{J}_i(0)| \leq \kappa_1 \frac{\|r\|_{\rho-2\delta}}{\epsilon^2}, \quad \forall\, t \in [0, \delta\epsilon^{-2}),$$

$i = 1, \dots, N$. We also know that $|x_i(t) - q_i(t)| \leq \epsilon\mu$ and $|x_i(t) + q_i(t)| \leq 2, \forall\, t \in [0, \delta\epsilon^{-2})$, so that from

$$|J_i(t) - J_i(0)| \leq |\tilde{J}_i(t) - J_i(t)| + |\tilde{J}_i(0) - J_i(0)| + |\tilde{J}_i(t) - \tilde{J}_i(0)|$$

we easily obtain

$$|J_i(t) - J_i(0)| \leq 8\epsilon\mu + 4\kappa_1 \frac{\|r\|_{\rho-2\delta}}{\epsilon^2}, \quad \forall\, t \in [0, \delta\epsilon^{-2}), \quad \forall\, i = 1, \dots, N. \qquad \square$$

Proof of Theorem 2. Lemma 5 is analogous to Lemmas 1 and 2 combined. The proof is exactly the same, with capital letters replacing small letters. In Lemma 6 we estimate times for which the trajectories stay in the domain of the canonical transformation, and in Lemma 7 we estimate the drift of the actions. $\square$

Lemma 5. *Let* $\rho > 5\delta > 0$ and $\Lambda_1, \Lambda_2 < 1$ *as in* (19). *Then properties* (1)–(3) *of Section* 4 *imply that there is a canonical transformation* $\Phi^{(\epsilon S_1, \epsilon^2 S_2)} : \mathcal{A}(D_{\rho-2\delta}) \to \mathcal{A}(D_\rho)$ *so that the transformed Hamiltonian* $H \circ \Phi^{(\epsilon S_1, \epsilon^2 S_2)} = H_0 + \epsilon^2 N_2 + R$, with N_2 *the quartic part of the normal form Hamiltonian of Proposition* 2 *and* R *the remainder i.e. the terms of* $\mathrm{O}(\epsilon^3)$. *The remainder satisfies*

$$\|R\|_{\rho-2\delta} \leq (\Lambda_1)^3 \tilde{\alpha}, \tag{28}$$

with $\tilde{\alpha}$ *the same as the* α *of Theorem* 1, *but with capital letters. Moreover consider the restriction* $\Phi^{(\epsilon S_1, \epsilon^2 S_2)}$: $\mathcal{A}(D_{\rho-4\delta}) \to \mathcal{A}(D_{\rho-2\delta})$. *Then*

$$|y_i - y_i \circ \Phi^{(\epsilon S_1, \epsilon^2 S_2)}| \leq \epsilon M, \quad \forall\, i = 1, \dots, 2N, \tag{29}$$

with M defined below.

Lemma 6. *Let* ρ, δ *as in Lemma* 5 *and let* $\tilde{y}_i = y_i \circ \Phi^{(\epsilon S_1, \epsilon^2 S_2)}$, $i = 1, \dots, 2N$ and $\tilde{y} = [\tilde{y}_1, \dots, \tilde{y}_{2N}]$. *Assume that* $\tilde{y}(0) \in D_{\tilde{\rho}}$, *with* $\tilde{\rho} \leq (4l_{\max} + 2)^{-3/2}(\rho - 5\delta)$. *Then*

$$\tilde{y}(t) \in D_{(4l_{\max}+2)^{3}2\tilde{\rho}+\delta}, \quad \forall\, t \in \left[0, \frac{\delta^2}{\sqrt{2}(2l_{\max}+1)\kappa_1 \|R\|_{\rho-2\delta}}\right).$$

Proof. Let $\tilde{B}_i = \tilde{I}_i^{1/2}$. Initially $\tilde{B}_i \le 2(2l_{\max}+1)^{1/2}$. Also, as in Lemma 3, for $t \in [0, T_0)$ for which $\tilde{B}_i \le 2(2l_{\max}+1)^{1/2}\tilde{\rho} + 2^{-1/2}(2l_{\max}+1)^{-1}\delta$ we have that

$$\left|\frac{\mathrm{d}\tilde{B}_i}{\mathrm{d}t}\right| \le \frac{1}{4(2l_{\max}+1)} \sup_{\tilde{B}_i \le b_0} \sum_{m=-l}^{l} \left(\left|\frac{\partial R}{\partial \tilde{q}_{[l,m]}}\right| + \left|\frac{\partial R}{\partial \tilde{p}_{[l,m]}}\right| \right),$$

where $b_0 = 2(2l_{\max}+1)^{1/2}\tilde{\rho} + 2^{-1/2}(2l_{\max}+1)^{-1}\delta$. Hence

$$\left|\frac{\mathrm{d}\tilde{B}_i}{\mathrm{d}t}\right| \le \frac{1}{4} \sup_{\tilde{y}\in D_{\rho-4\delta}} \left(\left|\frac{\partial R}{\partial \tilde{q}_{[l,m]}}\right| + \left|\frac{\partial R}{\partial \tilde{p}_{[l,m]}}\right| \right) \le \frac{\kappa_1}{2\delta}\|R\|_{\rho-2\delta}.$$

Then T_0 is at least $\delta^2(\sqrt{2}(2l_{\max}+1)\kappa_1\|R\|_{\rho-2\delta})^{-1}$. □

Lemma 7. *Let $\rho, \delta, \tilde{\rho}$ as in Lemmas 5 and 6, and ρ_2 as in Theorem 2. Let $y(0) \in D_{\rho_2}$ and assume that*

$$\epsilon M \le 3^{-1}(4l_{\max}+2)^{-3/2}(\rho - 5\delta), \qquad \frac{e^2\delta}{\sqrt{2}(2l_{\max}+1)\kappa_1\|R\|_{\rho-2\delta}} \le 1. \tag{30}$$

Then we have that $\forall\, i = 1, \dots, N$

$$|I_i(t) - I_i(0)| \le 4\epsilon M + \frac{\kappa_1\|R\|_{\rho-2\delta}}{2\delta\epsilon^2}, \quad \forall\, t \in [0, \delta\epsilon^{-2}).$$

Proof. The hypotheses in (30) imply that $\tilde{y}(t)$ and $y(t)$ stay within $D_{\rho-4\delta}$ for $t \in [0, \delta\epsilon^{-2})$ and we can estimate the drift of $\tilde{I}_i = (2l+1)^{-1}\sum_{m=-l}^{l}(\tilde{q}^2_{[l,m]} + \tilde{p}^2_{[l,m]})$ using

$$\left|\frac{d\tilde{I}_i}{dt}\right| \le \frac{\kappa_1}{2\delta}\|R\|_{\rho-2\delta}, \quad \forall\, t \in [0, T_0), \quad \text{with} \quad T_0 = \frac{\delta^2}{\sqrt{2}(2l_{\max}+1)\kappa_1\|R\|_{\rho-2\delta}}.$$

Therefore

$$|\tilde{I}_i(t) - \tilde{I}_i(0)| \le \frac{\kappa_1\|R\|_{\rho-2\delta}}{2\delta\epsilon^2}, \quad \forall\, t \in [0, T_0), \quad \forall\, i - 1, \dots, N.$$

Using the above and Lemma 5 we easily obtain the conclusion. □

The above stability estimates can be easily modified in the case where quartic or higher order terms of the Hamiltonian are considered. The quartic normal form will be modified in the way described in Section 4, and the constants of the truncated quartic normal form will persist. The main modification will be in the $\mathrm{O}(\epsilon^3)$ remainder terms, that will include $\mathrm{O}(\epsilon^4)$ corrections, leading to a slightly smaller stability time. Since most of the literature on normal form estimates deals with general real analytic non-linearities, the tools for modifying the estimates are quite standard (see e.g. [18]).

Also, for Galerkin truncations with more modes, the dependence of the estimates on the dimension N of the phase space is explicit, but it also enters through the size of H_1, and the constants c_1 and c_2 bounding the divisors corresponding to the cubic and quartic non-resonant terms. The norm of H_1 and the c_1, c_2 can be bounded above numerically for any desired $l_{\max}$.

6. Discussion

The arguments of Section 4 imply that the stability of near-monochromatic solutions (in the asymptotic sense we are considering) should persist for larger Galerkin truncations of Hamilton's equations (9) with the Hamiltonian

$H = H_0 + \epsilon H_1$, provided that property (3) of Section 4 is verified (numerically) for each truncation. Also, the addition of quartic terms $\epsilon^2 H_2$ or higher order terms to the Hamiltonian will not weaken the arguments of Section 4. Similarly, the addition of more modes or higher order non-linearity will not have any qualitative effect on the rigorous normal forms, although the domain of the transformations will shrink. It would also be interesting to investigate near-monochromatic motions from a more geometrical point of view, by calculating Liapunov orbits numerically (in the axisymmetric case), and looking at their stability and bifurcations.

Our argument for the reliability of the numerics was based on the quantitative agreement of results obtained through different local error tolerances, i.e. on the self consistency of the integration method (see [15]). It would be useful to use symplectic integrators, although we think that for the times we considered the results would be very close to what we obtained. A possible sign of discrepancy between symplectic and non-symplectic numerical integrators is the appearance of monotonic decreases or increases of the energy and actions in the non-symplectic integration (see [19] for an example). Such “numerical drifts” were observed, but were rare and of small size. Moreover we suppressed them by decreasing the error tolerance. We therefore think that, in the present problem, symplectic integration may be useful in significantly longer time integrations.

More efficient variants of the Galerkin method for calculations with higher order non-linearity and greater spatial resolution appear in [5,20,21] and elsewhere, for water waves on the plane (see also [22] for a similar problem in spherical geometry). However in these studies some form of artificial damping is needed to suppress blow-up and numerical instability in the higher modes, while in our simulations energy is conserved to high accuracy and boundedness of the solutions follows from conservation of energy and the form of the Hamiltonian (see [15]). Thus the approach we use here seems more appropriate for longer time integration and comparisons with the Hamiltonian theory.

Finally, the theory of the present work does not apply to the infinite dimensional case, since we cannot exclude the possibility of resonant terms that destroy some or all of the adiabatic invariants of the finite dimensional truncations and produce parametric instabilities of the approximate orbits. For 2-D water waves it was shown in [6,7] that the coefficients of analogous resonant terms vanish. We do not at present know whether such cancellations can occur on the sphere, e.g. for high frequency terms of axisymmetric solutions. Even in such a case it would be interesting to investigate the Diophantine equation of quartic resonance.

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Appendix A

We include three technical lemmas used in Section 5. Since the lemmas are variants of well known facts the proofs are omitted. The notation is that of the paragraph defining analytic functions and norms in Section 5.

Lemma A.1 (Cauchy estimate). *Let* $F \in \mathcal{A}(D_\rho)$ *and* $0 < \delta < \rho$. *Then* $\forall\, i = 1, \ldots, 2N$ *we have*

$$\left\| \frac{\partial F}{\partial x_i} \right\|_{\rho-\delta} \leq \frac{\kappa_1}{\delta} \|F\|_\rho, \quad with \quad \kappa_1 = e^{-1}\left(1 - \frac{\delta}{\rho}\right)^{-1}.$$

Lemma A.2 (Poisson bracket estimate). *Let* $F \in \mathcal{A}(D_\rho)$, $G \in \mathcal{A}(D_{\rho-\delta})$ *and* $0 < \delta, \delta' < \rho$. *Then*

$$\|[F, G]\|_{\rho-\delta-\delta'} \leq \frac{\tilde{c}}{\delta'(\delta+\delta')}\|F\|_{\rho-\delta}\|G\|_\rho, \quad with \quad \tilde{c} = 2N(\kappa_1)^2,$$

and κ_1 *as in Lemma* A.1.

Lemma A.3 (Convergence of the Lie-series). *Let* $S, H \in \mathcal{A}(D_\rho)$, *and define* $G_0 = H$, $G_r = 1/r[S, G_{r-1}]$ *for* $r \geq 1$. *Then*

$$\|G_r\|_{\rho-\delta} \leq B_r, \quad with \quad B_r = \left(\frac{\kappa_2}{\delta^2}\|S\|_\rho\right)^r \|H\|_\rho, \quad \kappa_2 = e^2\tilde{c},$$

and $\tilde{c}$ *as in Lemma* A.2. *If* $\kappa_2/\delta^2\|S\|_\rho < 1$ *then* $\exp(Ad_S H) \in \mathcal{A}(D_{\rho-\delta})$.

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