# Solitary waves in nematic liquid crystals

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**Abstract**

We study soliton solutions of a two-dimensional nonlocal NLS equation of Hartree-type with a Bessel potential kernel. The equation models laser propagation in nematic liquid crystals. Motivated by the experimental observation of spatially localized beams, see [CPA03], we show existence, stability, regularity, and radial symmetry of energy minimizing soliton solutions in $\mathbb{R}^2$. We also give theoretical lower bounds for the $L^2-$norm (power) of these solitons, and show that small $L^2-$norm initial conditions lead to decaying solutions. We also present numerical computations of radial soliton solutions. These solutions exhibit the properties expected by the infinite plane theory, although we also see some finite (computational) domain effects, especially solutions with arbitrarily small power.

## 1 Introduction

We study some basic properties of solitary waves in a nonlocal nonlinear Schrodinger (NLS) equation modeling the propagation of laser light in nematic liquid crystals. The model was proposed by Conti, Peccianti, and Assanto [CPA03], who also conducted experiments and found stable optical solitons with a two-dimensional transverse profile. Other physical systems modeled by this or related nonlocal NLS equations are discussed in [**?**]. The stabilization of solitons and related lack of blow-up in the model is due to the nonlocality of the nonlinear interaction, and was predicted by earlier theoretical works, see [GV80], [T86]. More recent experiments examined this effect in other physical systems [RCMSC05]. The regularizing effect of the nonlocal nonlinearity makes the liquid crystal system an interesting laboratory for studying two dimensional solitons, and there is considerable recent experimental and theoretical work on vortices [YZK05], [MSXK09], soliton interactions [CMMSW08], multicolor solitons [SS09] and other related coherent structures.

In the present work we show the existence, regularity, and radial symmetry of energy minimizing solitons and compute radial solitons numerically. We also give analytically lower bounds for the $L^2-$norm (power) of energy minimizing solitons of negative energy. These thresholds involve best constants for the Gagliardo-Nirenberg and Hardy-Littlewood inequalities. It is possible that initial conditions with positive energy decay. While we do not settle this issue here, we use a different line of reasoning to show that initial conditions with sufficiently small $L^2-$norm decay.

The original model of [CPA03] couples a Schrödinger equation for the evolution of the electric field amplitude to a nonlinear elliptic equation for the director field. The time variable is physically the distance along the optical axis. We here consider a common simplification of this model that leads to an NLS equation with a cubic Hartree-type nonlinearity on the plane, see [YZK05], [SS09]. The kernel of the Hartree nonlinearity is the two-dimensional Bessel potential (also known as a modified Bessel function). The particular NLS equation in two dimensions was discussed earlier in [KRZ86] where it was argued heuristically that it should have stable solitons, in contrast to the well known situation for the standard cubic NLS in two dimensions, where solitons are unstable and solutions can blow-up in finite time, see [W83]. Turytsin [T86] uses a Gagliardo-Nirenberg inequality and energy conservation to argue that the $H^1-$norm of the solutions should stay bounded for all times. A simpler energy argument for bounded Hartree kernels appears in [BKWR02], [KBNNWRE04]. A rigorous version of the energy argument is also implicit in the work of Ginibre and Velo [GV80], see also [C03], who consider cubic Hartree nonlinearities with more general kernels. that include the one studied here.

Soliton solutions are obtained by minimizing the Hamiltonian of the nonlocal NLS over $H^1$ functions with fixed $L^2-$norm (power). The existence of minimizers is shown here by a concentration-compactness argument. We note that P.L. Lions [L84a] considered related quartic functionals with Hartree kernels. We further use elliptic regularity and rearrangement inequalities to see that the minimizer is a smooth radially symmetric decreasing function (up to translation and global phase change). The existence of constrained energy minimizers assumes that the $L^2-$norm is above a certain threshold, and we give two explicit lower bounds for this threshold. The idea is to bound below a ratio involving the quartic and quadratic parts of the energy, and the power. A similar ratio appears in the work of Weinstein [W83] on the cubic NLS on the plane, although the present problem is closer to the situation in the discrete NLS [W99]. Our power threshold estimate here involves best constants for the Gagliardo-Nirenberg inequality, and we also note an alternative bound involving constants for the Gagliardo-Nirenberg and Hardy-Littlewood inequalities.

We also include a fixed point argument in a space-time Lebesgue space that shows that initial conditions with small power decay. This proof uses the Strichartz estimates for the free Schrödinger evolution, and is similar to the one of [CW89] for the cubic power NLS on the plane. The decay argument gives a third bound for the minimum $L^2-$norm of $H^1$ solitons. A similar combination of absence of blow-up and decay for small solutions was also seen in the discrete cubic NLS in the two dimensional integer lattice, see [SK05].

We also present numerical computations of positive, decreasing radial solitons. The numerical study uses a finite circular domain with Dirichlet boundary conditions, and solitons are computed using the method of [LMS13]. The numerical results are consistent with the existence of power threshold for negative energy solitons, but we also observe soliton-like solutions of arbitrarily small $L^2-$norm and positive energy. The existence of these small solutions can be explained by an abstract local bifurcation result, applied to the finite domain problem and its discretizations. We thus expect that only part of the calculated solution branch yields approximations to solitons of the $\mathbb{R}$ problem. The transition to the "spurious" part may involve collision with other solution branches, see [FKM97] for such a scenario in the discrete NLS. This problem is left for further work.

The paper is organized as follows. In Section 2 we state our main theoretical results for the planar ($\mathbb{R}^2$) problem. In Section 3 we review the main results used in the proofs. In Section 4 we prove the existence, regularity, and symmetry properties of the minimizing solitons. In Section 5 we prove the power threshold and small amplitude decay theorems for the planar problem. In Section 6 we present the numerical results, and interpret them using the planar theory and a local bifurcation argument for the finite domain problem. In Section 7 we briefly discuss some further problems.

## 2 Soliton solutions of the nematicon equation

We consider the single-color nematicon equation in $\mathbb{R}^2$

$$iu_t + \frac{1}{2}D\Delta u + 2A\theta u = 0, \tag{2.1}$$

$$-\Delta\theta + m^2\theta = \frac{A}{\nu}|u|^2, \tag{2.2}$$

with constants $D$, $A$, $\nu > 0$, e.g. compare with [SS09]. The variable $u$ represents the electric field envelope amplitude of an optical beam propagating through a nematic liquid crystal, while $\theta$ represents the angle of the director field of the liquid crystal.

The inhomogeneous elliptic equation (2.2) has a unique solution $\theta = G(|u|^2)$, with $G$ a linear operator of convolution type, so that system (2.1), (2.2) is equivalent to equation

$$iu_t + \frac{1}{2}D\Delta u + 2AG(|u|^2)u = 0. \tag{2.3}$$

Taking the Fourier transform of (2.2) we have

$$\hat{\theta}_k = \frac{\hat{f}_k}{|k|^2 + m^2}, \quad \text{with} \quad f = \frac{A}{\nu}|u|^2. \tag{2.4}$$

Thus $G$ is convolution with the inverse Fourier transform of $A\nu^{-1}(|k|^2 + m^2)^{-1}$, and we have

$$\theta(x) = G(|u|^2)(x) = \frac{A}{\nu}\int_{\mathbf{R}^2} K_0(m|x-y|)|u(y)|^2 d^2y \tag{2.5}$$

where $K_0$ is the modified Bessel function, see [B58], ch. III, or Bessel potential in $\mathbb{R}^2$ (up to constants), see [E98], p.186. $G$ is a bounded, self-adjoint operator in $L^2(\mathbb{R}^2, \mathbb{C})$.

The kernel $K_0 : \mathbb{R}^+ \to \mathbb{R}$ is positive, strictly decreasing and has the respective small and large $r$ asymptotics

$$K_0(r) = \frac{1}{2\pi}\left(-\log r + (\log 2 - \gamma)\right) + O(r^2), \quad \text{as} \quad r \to 0 \tag{2.6}$$

$$K_0(r) = \frac{1}{2\pi}\sqrt{\frac{\pi}{2r}}e^{-r}\left(1 + O(r^{-1})\right), \quad \text{as} \quad r \to \infty, \tag{2.7}$$

with $\gamma$ the Euler-Mascheroni constant, see [B58], ch. III-V. We also use the notation

$$K_{0,\mu}(r) = K_0(\mu r), \quad \mu, r > 0. \tag{2.8}$$

To avoid the singularity of $K_0$ at the origin, some authors have studied the nematicon system (2.3) using bounded kernels with a comparable fast decay at infinity, such as Gaussians, see [YLP06], [KBNNWRE04], [BKWR02]. Most of the qualitative results below seem to apply to these models as well, see [SBEK06] for some differences.

The nematicon equation (2.3) with $\theta = G(|u|^2)$, $G$ as above, can be also written formally as

$$u_t = -i\frac{\delta H}{\delta u^*}, \quad \text{with} \quad H = \int_{\mathbb{R}^2} \left( \frac{D}{2}|\nabla u|^2 - A|u|^2 G(|u|^2) \right), \tag{2.9}$$

i.e. $H$ is the *Hamiltonian* or *energy* of (2.3). Another conserved quantity of (2.3) is the *power* $P$, defined as

$$P(u) = \int_{\mathbb{R}^2} |u|^2, \tag{2.10}$$

see [SS99] for other conserved quantities.

In contrast to the two-dimensional cubic NLS whose solutions can blow up in finite time, see e.g. [GV79], [W83], the nonlocal analogue (2.3) with $G$ as in (2.5) has solutions that exist for all times.

To state the simplest long time existence result we let $L^p(\mathbb{R}^N) = L^p(\mathbb{R}^N; \mathbb{C})$, $H^s(\mathbb{R}^2) = H^s(\mathbb{R}^2; \mathbb{C})$, $s \in \mathbb{R}$. We then have:

**Theorem 2.1** *The initial value problem for (2.3) with $u(0) \in H^1(\mathbb{R}^2)$ has a unique solution $u \in C^0(\mathbb{R}; H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^2))$. Moreover, $||u(t)||_{H^1(\mathbb{R}^2)} \leq M_0$ for some $M_0 > 0$, for all $t \in \mathbb{R}$.*

The local existence follows from a standard fixed point argument. Global existence follows by a conservation of energy argument using the idea of [T86]. We give an abbreviated proof, since the argument is also implicit in [GV80], see also [C03], ch. 6.

The experimental observation of spatially localized solutions motivates the study of solutions of of (2.3) of the form $u(x,t) = e^{-i\omega t}\psi(x)$. Then $\psi$ must satisfy

$$\omega\psi = -\frac{D}{2}\Delta\psi - 2A\psi G(|\psi|^2). \tag{2.11}$$

To solve (2.11) we look for minimizers of $H$ over $H^1$ functions of constant $L^2$ norm. Such solutions also satisfy a nonlinear orbital stability property.

The operations of global phase change and translation are defined respectively as $(g_\phi\psi)(x) = e^{i\phi}\psi(x)$, $\phi \in \mathbb{R}$ arbitrary (and independent of $x$), and $(\tau_y\psi)(x) = \psi(x-y)$, $y \in \mathbb{R}^2$. They both take solutions of (2.11) to solutions of (2.11), and also leave the Hamiltonian $H$ and the other conserved quantities above invariant.

Let $\lambda > 0$ and define

$$I_\lambda = \inf\{H(u) : \ u \in H^1(\mathbb{R}^2), \ P(u) = \lambda\}. \tag{2.12}$$

A $u_* \in H^1(\mathbb{R}^2)$, $P(u_0) = \lambda$, satisfying $H(u_*) = I_\lambda$ is referred to as a *minimizer* or ground state (of $H$ at power $\lambda$). The set of minimizers of $H$ at power $\lambda$ is denoted as $\mathcal{M}_\lambda$.

**Theorem 2.2** *There exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ we have: (i) The set $\mathcal{M}_\lambda$ is nonempty. (ii) Any $u_* \in \mathcal{M}_\lambda$ is a $C^2$ solution of equation (2.11), and can be chosen to be real-valued, i.e. there exists $\phi \in \mathbb{R}$ such that $g_\phi u_*$ is real-valued. (iii) For any real $u_* \in \mathcal{M}_\lambda$ we can choose $y \in \mathbb{R}^2$ such that $\tau_y u_*$ is positive, radial, and strictly decreasing.*

The proof of (i) uses the concentration-compactness lemma of P.L. Lions, [L84a], [L84b]. In fact, these works already considered various classes of Hartree-type nonlinearities with integrability properties satisfied by the kernel $K_0$ in a somewhat different functional. The $C^2$ regularity of (ii) follows from the differentiability of the functional $H$, and elliptic regularity arguments (which apply to other critical points of $H$ as well). In view of the invariance of $H$, and $P$ under global phase change and translation, parts (ii) and (iii) state that minimizers are real up to global phase rotation, and radially symmetric, up to translation. Part (iii) uses rearrangement arguments. Conservation of $H$ along trajectories of (2.3) and (iv) imply the following well known property, see e.g. [C03], ch. 8.

**Corollary 2.3** *Let $\lambda > \lambda_0$. The set $\mathcal{M}_\lambda$ of solutions of (2.11) is orbitaly stable.*

The fact that the $H^1$ norm remains bounded, and the conservation of energy also imply that trajectories with initial conditions of negative energy cannot lead to decay.

**Corollary 2.4** *Consider a solution $u$ of (2.3) as in Theorem 2.1 with initial condition $u_0$ satisfying $H(u_0) < 0$. Such $u(t)$ can not satisfy $||u(t)||_{L^\infty(\mathbb{R}^2)} \to 0$ as $t \to \pm\infty$.*

The proof of Theorem 2.2 uses the fact that there exist states $u \in H^1$ satisfying $H(u) < 0$. We see that for $\lambda$ sufficiently large such states exist, and this is the origin of the condition $\lambda > \lambda_0$ in Theorem 2.2. The smallest possible value of such a threshold $\lambda_0$ is not known, but we show that it cannot be arbitrarily small.

**Proposition 2.5** *Let $v \in H^1(\mathbb{R}^2)$ satisfy $H(v) < 0$. Then $P(v) > \lambda_* = max\{\lambda_1, \lambda_2\}$, where*

$$\lambda_1 = \frac{D\nu}{2A^2}\frac{m^2}{(\kappa_{2,4,\frac{1}{2}})^4}, \quad \lambda_2 = \frac{D\nu}{2A^2}\frac{m^2}{(c_{2,2})^{\frac{1}{2}}(\kappa_{2,4,\frac{1}{2}})^4||K_0||_{L^1(\mathbb{R}^2)}}, \tag{2.13}$$

*where the constants $\kappa_{2,4,\frac{1}{2}}$, and $c_{2,2}$ appear in the Gagliardo-Nirenberg (Lemma 3.1), and Hardy-Littlewood (Lemma 3.10) inequalities respectively.*

Proposition 2.13 is obtained by examining the ratio between a product of the $L^2$ norms of $u$, $\nabla u$, and the quartic part $V$ of $H$, for $u \in H^1(\mathbb{R}^2)$, as in [W83], [W99]. The bounds $\lambda_1$, $\lambda_2$ follow from alternative upper estimates of $V$ that lead to a cancellation of the test functions from this ratio.

The above imply non-existence of negative energy solitons of small power. A more direct proof of the existence of a power threshold for solitons, i.e. any $H^1$ solutions of (2.11), is given by the following:

**Theorem 2.6** *Consider the solution $u$ of the initial value problem (2.3) with initial condition $u(0) \in H^1(\mathbb{R}^2)$, as in Theorem 2.1. There exists a $\lambda_3$ such that $||u(0)||_{L^2(\mathbb{R}^2)} \leq \lambda_3$ implies that $||u(t)||_{L^4(\mathbb{R}^2)} \to 0$ as $t \to +\infty$.*

We see below that the $L^4-$norm of solutions of Theorem 2.1 remains bounded for all times. By Theorem 2.6, if the power of the initial condition is sufficiently small, then the $L^4-$norm of the solution decays.

**Corollary 2.7** *There can not be solutions $\psi \in H^1(\mathbb{R}^2)$ of (2.11) with arbitrarily small $L^2-$norm.*

Theorem 2.6 uses the Strichartz estimates for dispersive decay of the free Schrödinger evolution on the plane, and the origin of the bound $\lambda_3$ is quite different. At present we have not compared or estimated numerically the bounds $\lambda_1$, $\lambda_2$, $\lambda_3$ (see [W83] for $\lambda_1$). It may also be that negative energy $H^1$ initial conditions lead to decay. In the cubic NLS case this follows from the virial identity and may require some extra assumptions see [SS99], [C03].

In Section 6 we also present numerical soliton profiles, obtained by looking for radial solutions of (2.11). We discretize (2.11) in a finite disc and use Dirichlet boundary conditions. The numerical study exhibits profiles with the properties expected by Theorem 2.2, and gives indirect evidence for the existence of a power threshold for negative energy solitons in Proposition 2.13. The finite domain evolution is not expected to have the decay properties of Theorem 2.6, and we also argue that the analogue of (2.11) should have solutions with arbitrarily small $^L2-$norm. Such solutions are also seen numerically.

The finite domain problem is left for future work. As we discuss in Sections 6 and 7, it could give numerical estimates of the threshold $\lambda_*$ of Proposition 2.13. Another open problem is the uniqueness of the minimizers, modulo translations and global phase rotations.

## 3 Some preliminary results

We state some basic results that will be used in the following sections.

**Lemma 3.1** *(Gagliardo-Nirenberg inequality, dimension $N = 2$. See [C03], p.9.) For all $u \in H^1(\mathbb{R}^2))$ we have*

$$||u||_{L^p(\mathbb{R}^2)} \leq \kappa_{2,p,\alpha}||\nabla u||^{\alpha}_{L^2(\mathbb{R}^2)}||u||^{1-\alpha}_{L^2(\mathbb{R}^2)}, \quad 1 \leq p \leq \infty, \quad \alpha \in [0,1), \quad p = \frac{2}{1-\alpha}. \tag{3.1}$$

*The constant $\kappa_{2,p,\alpha}$ only depends on $p$, and $\alpha$.*

To prove Theorem 2.2 (i) we solve a minimization problem in unbounded domains. The main technical tool is the P.L. Lions' concentration compacteness Lemma below.

**Lemma 3.2** *(Concentration-compactness Principle. See [L84a], [C03], ch. 1.7). If $\lambda > 0$ and $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence of $H^1$ with $P(u_k) \equiv ||u_k||^2_{L^2(\mathbb{R}^N)} = \lambda$, then there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ for which one of the following properties holds:*

*1) (compactness) There exists a sequence* $\{x_j\}_{j\in\mathbb{N}}$ *and* $u \in H^1(\mathbb{R}^N)$ *such that* $\{\tau_{x_{k_j}} u\}_{j\in\mathbb{N}}$ *converges to* $u$ *in* $L^p(\mathbb{R}^N)$, $\forall p \in [2, 2N(N-2)^{-1})$ *if* $N > 1$ ($\forall p \in [2,\infty]$, *if* $N = 1$).

*2) (vanishing)* $\{u_{k_j}\}_{j\in\mathbb{N}}$ *converges to the origin in* $L^p(\mathbb{R}^N)$, $\forall p \in (2, 2N(N-2)^{-1})$ *if* $N > 1$ ($\forall p \in (2,\infty]$, *if* $N = 1$).

*3) (splitting) There exists* $0 < \mu < \lambda$ *such that for every* $\varepsilon > 0$ *there exists* $j_0 \geq 0$ *and two sequences* $\{u'_j\}_{j\in\mathbb{N}} \subseteq H^1(\mathbb{R}^N)$ *and* $\{u''_j\}_{j\in\mathbb{N}} \subseteq H^1(\mathbb{R}^N)$ *with compact disjoint supports, such that for* $j \geq j_0$

$$||u'_j||_{H^1(\mathbb{R}^N)} + ||u''_j||_{H^1(\mathbb{R}^N)} \leq 4 \sup_{j\in\mathbb{N}} ||u_{k_j}||_{H^1(\mathbb{R}^N)}, \tag{3.2}$$

$$||u_{k_j} - u'_j - u''_j||_{L^2(\mathbb{R}^N)} \leq \varepsilon, \tag{3.3}$$

$$\left| \int_{\mathbb{R}^N} |u'_j(x)|^N \, dx - \mu \right| \leq \varepsilon, \tag{3.4}$$

$$\left| \int_{\mathbb{R}^N} |u''_j(x)|^2 \, dx + \mu - \lambda \right| \leq \varepsilon, \tag{3.5}$$

$$||\nabla u'_j||_{L^2(\mathbb{R}^N)} + ||\nabla u''_j||_{L^2(\mathbb{R}^N)} \leq ||\nabla u_{k_j}||_{L^2(\mathbb{R}^N)} + \varepsilon. \tag{3.6}$$

*Moreover,* $dist(supp(u'_j), supp(u''_j)) > 2\varepsilon^{-1}$.

**Remark 3.3** *In the case of splitting of Lemma 3.2 (i.e. case 3)* $u'_j$, $u''_j$ *can be chosen to be of the form* $u'_j(x) = \rho(x - x_j)u_m(x)$, $u''_j(x) = \theta(x - x_j)u_m(x)$, *where* $\{x_j\}_{j\in\mathbb{N}}$ *is a sequence of points in* $\mathbb{R}^N$, *and the functions* $\rho$, $\vartheta : \mathbb{R}^N \to [0,1]$ *are* $C^\infty$, *radial and satisfy*

$$\begin{array}{ll} (i) & |\rho'(|x|)|, |\vartheta'(|x|)| < \epsilon, \quad \forall x \in \mathbb{R}^N, \\ (ii) & \rho(x) = 1, \quad \text{if} \quad |x| < t_1; \quad \rho(x) = 0, \quad \text{if} \quad |x| \geq t_1 + 2\epsilon^{-1}, \\ & \vartheta(x) = 1, \quad \text{if} \quad |x| > t_2; \quad \vartheta(x) = 0, \quad \text{if} \quad |x| \leq t_2 - 2\epsilon^{-1}, \end{array}$$

*where* $0 < t_1 < t_2$, $t_2 - t_1 > 6\epsilon^{-1}$. *The above imply that* $supp\rho \cap supp\vartheta = \emptyset$, $dist(supp\rho, supp\vartheta) > 2\epsilon^{-1}$. *Moreover* $1 - \rho(x - x_j) - \vartheta(x - x_j) \geq 0$, $\forall x$, $x_j \in \mathbb{R}^N$. *More details can be found in [C03] sec. 1.7, [ZGJT01], Lemma 6.1.*

The proof that minimizers are radial uses some results on rearrangements in $\mathbb{R}^N$. A Lebesgue measurable function $f : \mathbb{R}^M \to \mathbb{R}$ is *decreasing at infinity* if the Lebesgue measure $\mu_L(U_t(f))$ of $U_t(f) = \{x \in \mathbb{R}^N : \ |f(x)| > t\}$ is finite, $\forall t \geq 0$. Let $A \subset \mathbb{R}^N$ be a Borel set. Then $\overline{A}$ is the $N-$ball around the origin satisfying $\mu_L(\overline{A}) = \mu_L(A)$. Also if $\chi_A$ is the characteristic function of $A$, then $\overline{\chi}_A = \chi_{\overline{A}}$. The *radial symmetrization* $\overline{f}$ of $f$ decreasing at infinity is defined as $\overline{f}(x) = \int_{\mathbb{R}} \overline{\chi}_{U_t(f)}(x) \, dt$. We the have the following:

**Lemma 3.4** *(Riesz's rearrangement inequality. See [LL01], ch. 3) Let* $f$, $g$, $h : \mathbb{R}^N \to \mathbb{R}$ *be nonegative and decreasing at infinity. Then*

$$\int_{\mathbb{R}^N} f(x) \, g(x-y) \, h(y) \, dx \, dy \leq \int_{\mathbb{R}^N} \overline{f}(x) \, \overline{g}(x-y) \, \overline{h}(y) \, dx \, dy. \tag{3.7}$$

*If $g$ is radial and strictly decreasing, equality holds only if both $f(x-z)$, $g(x-z)$ are radial for some $z \in \mathbb{R}^N$.*

**Lemma 3.5** *(Polya-Szegö inequality. See [LL01], p.189) Let $f \in H^1(\mathbb{R}^N)$ be nonnegative and decreasing at infinity. Then*

$$\int_{\mathbb{R}^N} |\nabla \overline{f}(x)|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla f(x)|^2 \, dx. \tag{3.8}$$

*Equality holds only if $f(x-z)$ is radial for some $z \in \mathbb{R}^N$.*

Also, in the estimation of the power threshold $\lambda_0$ we use two results on maximal functions. Recall that the *maximal function* $\mathcal{M}f$ of $f \in L^1_{loc}(\mathbb{R}^N;\mathbb{R})$ as

$$\mathcal{M}f(x) = \sup_{r>0} |B_r(x)|^{-1} \int_{B_r(x)} |f(y)| dy, \tag{3.9}$$

where $|B_r(x)|$ is the $N-$volume of the $N-$ball of radius $r$ centered at $x \in \mathbb{R}^N$. We then have:

**Lemma 3.6** *(Hardy-Littlewood inequality. See [LP09], ch. 2) Let $1 < p < \infty$. Then*

$$\int_{\mathbb{R}^N} |\mathcal{M}f(x)|^p \, dx \leq c_{N,p} \int_{\mathbb{R}^N} |f(x)|^p \, dx. \tag{3.10}$$

**Lemma 3.7** *(See [LP09], ch. 2) Let $\mathcal{K} \in L^1(\mathbb{R}^N;\mathbb{R})$ be radial, positive, and decreasing, and $f \in L^1(\mathbb{R}^N;\mathbb{R})$. Then*

$$\sup_{\tau>0} \left| \int_{\mathbb{R}^N} \tau^{-N} \, \mathcal{K}(\tau^{-1}(x-y)) \, f(y) \, dy \right| \leq ||\mathcal{K}||_{L^1(\mathbb{R}^N)} \, \mathcal{M}f(x), \quad \forall x \in \mathbb{R}^N. \tag{3.11}$$

We recall the Strichartz estimates for the free Schrödinger evolution operator $S$, defined by letting $u(t) = S(t)\phi = e^{it\Delta}\phi$ be the unique solution of $u_t = i\Delta u$ with initial condition $\phi \in L^2(\mathbb{R}^2)$. A pair of indices $(q,r)$ is *admissible* (in dimension $N=2$) if $1/q + 1/r = 1/2$, with $r \in [0,\infty)$. The following inequalities involving the space-time Lebesgue spaces can be found in [C03], ch. 2.3.

**Lemma 3.8** *(Strichartz estimates, dimension $N=2$.) Let $(q,r)$ be an admissible pair, and let $S$ be the free Schrödinger evolution operator $S$ in $L^2(\mathbb{R}^2)$, as above. Let $[0,T]$ be a closed interval in $\mathbb{R}$, with $T \in [0,\infty]$. Then*

$$||S(\cdot)\phi||_{L^q([0,T],L^r(\mathbb{R}^2))} \leq C_1 ||\phi||_{L^2(\mathbb{R}^2)}. \tag{3.12}$$

*Also, let $(\gamma,\rho)$ be an admissible pair and assume that*

$$f \in L^{\gamma'}([0,T], L^{\rho'}(\mathbb{R}^2)), \tag{3.13}$$

*with the notation $p' = (1-1/p)^{-1}$. Then*

$$|| \int_0^{(\cdot)} S((\cdot)-s) f(s) ds ||_{L^q([0,T],L^r(\mathbb{R}^2))} \leq C_2 ||f||_{L^{\gamma'}([0,T],L^{\rho'}(\mathbb{R}^2))}. \tag{3.14}$$

*The constants $C_1$, $C_2$ (for $N=2$) only depend on $r$, and $r$, $\rho$ respectively, i.e. are independent of $T$.*

**Remark 3.9** *The time-rescaled operator $\tilde{S} = S(\alpha t)$, $\alpha = D/2 > 0$, satisfies estimates (3.12), (3.14) with $C_1$, $C_2$ replaced by $\tilde{C}_1 = \alpha^{-1/q} C_1$, $\tilde{C}_2 = \alpha^{-1/2-1/\gamma'} C_2$ respectively.*

# 4 Existence and symmetry of ground states

We use the following notation. Let $X = H^1(\mathbb{R}^2)$, its dual $X^* = H^{-1}(\mathbb{R}^2)$. Let $Z$, $Y$ be real or complex Banach spaces. Then $L(Z, Y)$ denotes the set of linear bounded operators from $Z$ to $Y$. Also, for $f : Z \to Y$ Fréchet differentiable, $Df \in L(Z, Y)$ denotes the Fréchet derivative of $f$.

*Proof of Theorem 2.1:*

*Proof.* To show local existence in $C^0([0, T], X)$, for some $T > 0$, we use Picard iteration for the integral form of the variational equation, i.e. we let

$$[\mathcal{G}(u)](t) = \tilde{S}(t)\phi + [F(u)](t), \tag{4.1}$$

with $\tilde{S}(t) = e^{it\frac{D}{2}\Delta}$, and

$$[F(u)](t) = -i2A \int_0^t \tilde{S}(t-s)\theta(s)u(s)ds, \quad \theta = G(|u|^2) \tag{4.2}$$

and solve $u = \mathcal{G}(u)$. The linear part generates an isometry in $X$ and it is enough to check that the nonlinearity of (2.3) is Lipschitz in $X$. We apply the Gagliardo-Nirenberg inequality (3.1) repeatedly (see the global existence argument below) and omit the details. Taking the time derivative of the integral equation we see that the time derivative $u_t$ belongs to $X^*$. By $u \in C^0([0, T], X)$ we easily check that $u_t$ is also continuous in $t$. Note also that the time $T$ depends only on $||u_0||_{H^1(\mathbb{R}^2)}$.

The local existence theory implies the conservation of $H$, and $P$ for solutions $u(t) \in C^0(\mathbb{R}; H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^2))$, see e.g. [GV80]. By Hölder's inequality the quartic part of $H$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^2} (K_{0,m} * |u|^2)(x)|u(x)|^2 dx &\leq ||K_{0,m} * |u|^2||_{L^\infty(\mathbb{R}^2)} ||u||^2_{L^2(\mathbb{R}^2)} && (4.3)\\ &\leq \lambda ||K_{0,m}||_{L^q(\mathbb{R}^2)} ||u||^2_{L^{2p'}(\mathbb{R}^2)}, \quad q > 1, \quad p' = \frac{q}{q-1}, && (4.4) \end{aligned}$$

where $\lambda = P(u) = ||u||^2_{L^2(\mathbb{R}^2)}$ remains constant along the trajectory. By the small and large $|x|$ asymptotics of $K_0$ in (2.6), (2.7) respectively, the $L^q$ norm of $K_0$ is finite, $\forall q > 1$. By the Gagliardo-Nirenberg inequality (3.1) we then have

$$\begin{aligned} \int_{\mathbb{R}^2} (K_{0,m} * |u|^2)(x)|u(x)|^2 dx &\leq \lambda \kappa_{2,2p',\beta} m^{-\frac{2}{q}} ||K_0||_{L^q(\mathbb{R}^2)} ||\nabla u||^{2\beta}_{L^2(\mathbb{R}^2)} ||u||^{2(1-\beta)}_{L^2(\mathbb{R}^2)} && (4.5)\\ &= \lambda^{2-\beta} \kappa_{2,2p',\beta} m^{-\frac{2}{q}} ||K_0||_{L^q(\mathbb{R}^2)} ||\nabla u||^{2\beta}_{L^2(\mathbb{R}^2)}, && (4.6) \end{aligned}$$

with $q > 1$, $\beta = q^{-1}$.

We can choose any $q > 1$, so that $\beta < 1$. Then

$$H(u) \geq \frac{D}{2}(||\nabla u||^2_{L^2(\mathbb{R}^2)} - \lambda) - \lambda^{2-\beta} \kappa_{2,2p',\beta} m^{-\frac{2}{q}} A^2 \nu^{-1} ||K_0||_{L^q(\mathbb{R}^2)} ||\nabla u||^{2\beta}_{L^2(\mathbb{R}^2)}, \tag{4.7}$$

with $\beta < 1$, $\quad q > 1$.

By the conservation of $H$, $P$, (4.7) implies that $||u(t)||_{H^1(\mathbb{R}^2)}$ must remain bounded for all $t \in [0, T]$ by a constant $M_0$ that depends only on $||u_0||_{H^1(\mathbb{R}^2)}$. Then the existence interval can be extended to $\mathbb{R}$, and $||u(t)||_{H^1(\mathbb{R}^2)} \leq M_0$, $\forall t \in \mathbb{R}$. $\square$

The proof of Theorem 2.2 follows from the lemmas below. Lemma 4.1 shows that a minimizing sequence is bounded in $H^1$ and that the concentrated compactness hypotheses apply. Lemma 4.3 shows that the infimum $I_\lambda$ is strictly negative, provided the $\lambda$ is sufficiently large. This eliminates the vanishing scenario. Lemma 4.5 rules out the splitting scenario.

**Lemma 4.1** *$I_\lambda > -\infty$. Moreover, any minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \in H^1(\mathbb{R}^2)$, $P(u_n) = \lambda$, for $H$ satisfies $||u_n||_{H^1(\mathbb{R}^2)} \leq M$, $\forall n \in \mathbb{N}$.*

*Proof.* Clearly, $I_\lambda < \infty$. Also each element $u_n$ of the minimizing sequence satisfies inequality (4.7). This implies that if $||\nabla u_n||_{L^2(\mathbb{R}^2)}$ diverges, then $I_\lambda = \infty$, a contradiction. Thus $||u_n||_{H^1(\mathbb{R}^2)} \leq M$, $\forall n \in \mathbb{N}$. By (4.7) we then have a finite lower bound

$$H(u_n) \geq -\frac{1}{2}\lambda^{2-\beta}\kappa_{2,2p',\beta}m^{-\frac{2}{q}}A^2\nu^{-1}||K_0||M^{2\beta}, \quad \forall n \in \mathbb{N}. \tag{4.8}$$

□

**Lemma 4.2** *(Subadditivity) $I_{\lambda_1+\lambda_2} < I_{\lambda_1} + I_{\lambda_2}$.*

The proof relies on the fact that the negative part of $H(u)$ is homogeneous quartic in $u$, and is thus similar to the proof for power nonlinearities.

**Lemma 4.3** *There exists a $\lambda_0 > 0$ such that $I_\lambda < 0$, $\forall \lambda > \lambda_0$.*

*Proof.* We consider a radial trial function $v(r) = af(\frac{r}{s}) \in H^1(\mathbb{R}^2)$, $r = |x|$, and show that we can adjust the parameters $a$, $s > 0$ to make $H(v) < 0$, provided $||v||_{L^2(\mathbb{R}^2)}$ is sufficiently large. We have

$$||v||^2_{L^2(\mathbb{R}^2)} = 2\pi a^2 s^2 I_2, \quad ||\nabla v||^2_{L^2(\mathbb{R}^2)} = 2\pi a^2 I_{22}, \quad V(v) = 2\pi A\nu^{-1}a^4s^4A\nu^{-1}I_4(s), \tag{4.9}$$

where $V$ is the quartic part $H$, and

$$I_2 = \int_0^\infty rf^2(r)dr, \quad I_{22} = \int_0^\infty r(f'(r))^2dr, \tag{4.10}$$

$$I_4(s) = \int_0^\infty \int_0^\infty I(sr_1, sr_2)r_1r_2f^2(r_1)f^2(r_2)dr_1\,dr_2, \tag{4.11}$$

$$I(sr_1, sr_2) = 2\int_0^\pi K_0(sm(r_1^2 + r_2^2 - 2r_1r_2\cos\theta)^{1/2})d\theta. \tag{4.12}$$

We eliminate the parameter $a$ in (4.9) using $||v||^2_{L^2(\mathbb{R}^2)} = 2\pi a^2s^2I_2 = \lambda$. Then

$$H(v) = \frac{DI_{22}}{2s^2I_2}\lambda - \frac{A^2}{2\pi\nu I_2^2}I_4(s)\lambda^2. \tag{4.13}$$

Fix a function $f$, and $s$. Then $I_2$, $I_{22}$ are positive constants. By the positivity of $G$, $I_4(s)$ is also a positive constant. By (4.13), $H(v) < 0$ for $\lambda$ sufficiently large. □

Equation (4.13) with an arbitrary choice of $f$, $s$ also leads to a rough estimate for $\lambda_0$. For instance, [SS09] calculated some of the integrals for the test function $f(r) = \mathrm{sech}^2 r$. We see in the next section that we can estimate $\lambda_0$ above by a quantity that is independent of $s$ and $f$.

To eliminate the vanishing scenario of Lemma 3.2 we observe that by (4.3), $p' > 1$, vanishing for the minimizing sequence implies the vanishing of the quartic part of $H$. Then $I_\lambda \leq 0$, contradicting Lemma 4.3 for $\lambda > \lambda_0$. A preliminary step is:

**Lemma 4.4** *Let $u$, $v \in H^1(\mathbb{R}^2)$ satisfying $dist(supp(u), supp(v)) > 2\varepsilon^{-1}$. Let $G$ be as in (2.5). Then*

$$\int_{\mathbb{R}^2} |u(x)|^2 (G(|v|^2))(x) dx \leq \mu ||u||^2_{L^2(\mathbb{R}^2)} ||v||^2_{L^2(\mathbb{R}^2)} e^{-\frac{\delta}{\varepsilon}}, \tag{4.14}$$

*with $\mu$, $\delta > 0$ depending only on $G$.*

*Proof.* We use the fact that for $r \leq R$, $K_{0,m}(r) \leq \mu e^{-\delta r}$ for some $\mu$, $\delta > 0$, by (2.7). Let $S_u = \mathrm{supp}(u)$, $S_v = \mathrm{supp}(v)$, and assume $\mathrm{dist}(S_u, S_v) > 2\epsilon^{-1}$ with $2\varepsilon^{-1} > R$. Then

$$\begin{aligned} \int_{\mathbb{R}^2} |u|^2 G(|u|^2) &= A\nu^{-1} \int_{S_u} |u(x)|^2 \left( \int_{S_v} K_{0,m}(x-y) |v(y)|^2 \, dy \right) dx \\ &\leq A\nu^{-1} (\sup\nolimits_{x \in S_u, y \in S_v} K_{0,m}(x-y)) \int_{S_u} |u(x)|^2 \left( \int_{S_v} |v(y)|^2 \, dy \right) dx. \end{aligned} \tag{4.15}$$

Since $K_{0,m}$ is decreasing, (4.16) becomes

$$\int_{\mathbb{R}^2} |u|^2 G(|u|^2) \leq e^{-2\delta\varepsilon^{-1}} A\nu^{-1} \int_{S_u} |u(x)|^2 \left( \int_{S_v} |v(y)|^2 \, dy \right) dx, \tag{4.16}$$

and we immediately obtain the lemma. □

**Lemma 4.5** *Let $\{u_n\}_{n \in \mathbb{N}} \in H^1(\mathbb{R}^2)$, $P(u_n) = \lambda$, be a minimizing sequence for $H$. Then splitting does not occur.*

*Proof.* We consider the splitting scenario: $\forall \epsilon > 0$ there exist an $m_0 > 0$, and subsequence $\{u_m\}_{m \in \mathbb{N}}$ such that $m > m_0$ implies that $u_m = u'_m + u''_m + h_m$, with $u'_m$, $u''_m$ as in Lemma 3.2. We then have

$$H(u_m) = H(u'_m) + H(u''_m) + R_m, \tag{4.17}$$

where $R_m = R_m^1 + R_m^2 + R_m^3$, and

$$
\begin{aligned}
R_m^1 &= D\int_{\mathbb{R}^2} \mathrm{Re}[(\nabla u_m')^* \cdot \nabla u_m''] \\
&\quad -A\int_{\mathbb{R}^2}[|u_m'|^2 G(2\mathrm{Re}((u_m')^* u_m'')) + |u_m''|^2 G(2\mathrm{Re}((u_m')^* u_m''))] && (4.18)\\
R_m^2 &= D\int_{\mathbb{R}^2} \mathrm{Re}[(\nabla u_m)^* \cdot \nabla h_m] && (4.19)\\
R_m^3 &= -A\int_{\mathbb{R}^2}(|u_m'|^2 + |u_m''|^2) + |h_m'|^2) G(|h_m'|^2) \\
&\quad -A\int_{\mathbb{R}^2}(|u_m'|^2 + |u_m''|^2 + |h_m'|^2) G(2\mathrm{Re}((u_m')^* h_m')) \\
&\quad -A\int_{\mathbb{R}^2}(|u_m'|^2 + |u_m''|^2 + |h_m'|^2) G(2\mathrm{Re}((u_m'')^* h_m')) \\
&\quad -A\int_{\mathbb{R}^2}|h_m'|^2 G(|u_m'|^2 + |u_m''|^2). && (4.20)
\end{aligned}
$$

The overlap of $u_m'$, $u_m''$ vanishes by Lemma 3.2, and the term on the first line of $R_m^1$ vanishes. The second line is bounded by Lemma 4.4 and is thus of $O(\epsilon)$.

To estimate $R_m^3$ we note that by Lemma 3.2 we have $||h_m||_{L^2(\mathbb{R}^2)} \leq \epsilon$, and $||\nabla h_m||_{L^2(\mathbb{R}^2)} \leq 5||u_m||_{H^1(\mathbb{R}^2)}$ ($\leq 5M$). We then apply Hölder's inequality and Lemma 3.1 to see that each term that is proportional to $h_m$ is bounded by a term $C_M \epsilon^\nu$, with $\nu > \frac{1}{2}$, and $C_M$ a constant that depends only on $M$.

The integrand in $R_m^2$ is proportional to $\nabla h_m$. This is not necessarily small, however it can be written as small plus nonegative: using Remark 3.3.1 we may write $h_m = (1 - \rho_m + \vartheta_m)u_m$, where $\rho_m(x) = \rho(|x - x_m|)$, $\vartheta_m(x) = \vartheta(|x - x_m|)$. Then

$$
R_m^2 = A\int_{\mathbb{R}^2}(-\mathrm{Re}[(\nabla u_m)^* \cdot (\nabla\rho_m + \nabla\vartheta_m)u_m]) + A\int_{\mathbb{R}^2}(1 - \rho_m + \vartheta_m)|\nabla u_m|^2). \tag{4.21}
$$

Using the bounds on $\partial_r \rho_m$, $\partial_r \vartheta_m$ from Remark 3.3.1, the first integral, denoted by $\tilde{R}_m^2$, is estimated as

$$
\begin{aligned}
|\tilde{R}_m^2| &\leq D\int_{\mathbb{R}^2}|\mathrm{Re}[(\nabla u_m)^*(\nabla\rho_m + \nabla\vartheta_m)u_m]| \\
&\leq D||\nabla u_m||_{L^2(\mathbb{R}^2)}||(\nabla\rho_m + \nabla\vartheta_m)u_m||_{L^2(\mathbb{R}^2)} \\
&\leq D||\nabla u_m||_{L^2(\mathbb{R}^2)}||\nabla\rho_m + \nabla\vartheta_m||_{L^\infty(\mathbb{R}^2)}||u_m||_{L^2(\mathbb{R}^2)} \leq \tilde{C}\epsilon, && (4.22)
\end{aligned}
$$

with $\tilde{C}$ depending on $M$. The second integral in (4.21) is nonegative.

Also, by Lemma 3.3, $P(u_m') = \lambda_1 + \beta_m'$, $P(u_m'') = \lambda_2 + \beta_m''$, with $\lambda_1 + \lambda_2 = \lambda$, $|\beta_m'|$, $|\beta_m''| < \epsilon$. Letting

$$
\tilde{u}_m' = \frac{\sqrt{\lambda_1}u_m'}{\sqrt{\lambda_1 + \beta_m'}}, \quad \tilde{u}_m'' = \frac{\sqrt{\lambda_2}u_m''}{\sqrt{\lambda_2 + \beta_m''}}, \quad r_m' = H(u_m') - H(\tilde{u}_m'), \quad r_m'' = H(u_m'') - H(\tilde{u}_m'') \tag{4.23}
$$

we easily check that $|r_m'|$, $|r_m''| \leq C\epsilon$, with $C$ depending on $M$ only.

Collecting the above we have

$$H(u_m) \geq H(\tilde{u}'_m) + H(\tilde{u}''_m) + \tilde{R}_m, \quad \text{with} \quad |\tilde{R}_m| \leq \tilde{C}\epsilon^{1/2}, \tag{4.24}$$

and $\tilde{C}$ depending on $M$. Taking $\epsilon$ sufficiently small and using strict subadditivity we have $H(u_m) > I_\lambda(z)$, a contradiction. $\square$

Therefore by Lemma 3.2 we have a subsequence $\{u_j\}_{j\in\mathbb{N}}$ of the minimizing sequence that converges to some $u \in H^1(\mathbb{R}^2)$ in $L^p(\mathbb{R}^2)$, $p \in [2,\infty)$. Then $\{V(u_j)\}_{j\in\mathbb{N}}$ converges to $V(u)$, where $V$ is the quartic part of $H$. Also, $\{u_j\}_{j\in\mathbb{N}}$ convergences weakly to $u$ in $H^1(\mathbb{R}^2)$, and by the weak lower-semicontinuity of the square of the norm we see that $\{H(u_j)\}_{j\in\mathbb{N}}$ converges to $H(u) = I_\lambda$. We then have convergence of the $H^1-$norms and finally that $\{u_j\}_{j\in\mathbb{N}}$ converges to $u$ in $H^1(\mathbb{R}^2)$.

Thus the existence of the minimizer is complete.

To prove Corollary 2.4, we estimate the quartic term of $H$, essentially as in the proof of the fact that minimizing sequences cannot vanish. In particular, using Hölder, Gagliardo-Nirenberg, and the boundedness of $G$ in $L^2$,

$$\begin{aligned}
\int_{\mathbb{R}^2} G(|u|^2)|u|^2 &\leq ||u||_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} G(|u|^2)|u| \\
&\leq \mathcal{C}_G ||u||_{L^\infty(\mathbb{R}^2)} |||u|^2||_{L^2(\mathbb{R}^2)} ||u||_{L^2(\mathbb{R}^2)} \\
&\leq \mathcal{C}_G \kappa_{2,4,\frac{1}{2}} \lambda ||\nabla u||_{L^2(\mathbb{R}^2)} ||u||_{L^\infty(\mathbb{R}^2)},
\end{aligned} \tag{4.25}$$

where $\mathcal{C}_G$ is the operator norm of $G$ in $L^2$. The fact that the $H^1$ norm of the solutions remains bounded implies that if $||u(t)||_{L^\infty(\mathbb{R}^2)}$ vanishes as $t \to \infty$ then the quartic part of $H$ also vanishes as $t \to \infty$, and we have a contradiction with the conservation of energy and the assumption of negative energy at $t = 0$.

We now see the other basic properties of minimizers.

**Lemma 4.6** *$H$, $P$ are $C^1$ functionals in $X$.*

*Proof.* To check that $H$ is Fréchet differentiable in $X$, and that $DH \in L(X;\mathbf{R}) = X^* = H^{-1}(\mathbb{R}^2)$ is the right hand side of (2.11) it suffices to show that $H'(\psi) = -\frac{D}{2}\Delta\psi - 2A\psi G(|\psi|^2)$ is an element of $X^*$, $\forall \psi \in X$, and that $H'(\psi)$ satisfies $H(\psi + w) - H(\psi) - (H'(\psi), w) = o(||w||_X)$, as $||w||_X \to 0$, $\forall \psi \in X$, i.e. we show that $DH = H'$. These follow by repeatedly applying Hölder and Gagliardo-Nirenberg. The continuity follows in the same way. The case of $P$ is clear. $\square$

**Lemma 4.7** *Let $u_* \in H^1(\mathbb{R}^2)$ satisfy $H(u_*) = I_\lambda$, $\lambda > \lambda_0$. Then $u_*$ is a $C^2$ solution of (2.11).*

*Proof.* By Lemma 4.6 the minimizer $\psi = u_* \in H^1(\mathbb{R}^2)$ satisfies (2.11) in $H^{-1}$. By Sobolev embedding $\psi \in H^1(\mathbb{R}^2)$ implies $\psi \in L^4(\mathbb{R}^2)$ and $|\psi|^2 \in L^2(\mathbb{R}^2)$. Then $\theta = G(|\psi|^2) \in H^2(\mathbb{R}^2)$ by (2.4), hence $\theta \in L^\infty(\mathbb{R}^2)$ by Sobolev embedding. Then $\theta\psi \in L^2(\mathbb{R}^2)$, and we have $\psi = (-\Delta + 1)^{-1}[\theta\psi + (\omega+1)\psi] \in H^2(\mathbb{R}^2)$ by (2.11). We use $\psi \in H^2(\mathbb{R}^2)$, $\theta \in H^2(\mathbb{R}^2)$ and Hölder to estimate $\nabla(\theta\psi)$. We see that $\theta\psi \in H^1(\mathbb{R}^2)$, and use (2.11) to show $\psi \in H^3(\mathbb{R}^2)$. We repeat the argument for the second partial derivatives of $\theta\psi$. We show that $\psi \in H^3(\mathbb{R}^2)$, $\theta \in H^2(\mathbb{R}^2)$ imply $\theta\psi \in H^2(\mathbb{R}^2)$, hence $\psi \in H^4(\mathbb{R}^2)$ by (2.11). Hence the restriction of $\psi$ to any bounded domain with smooth boundary $\Omega \subset \mathbb{R}^2$ belongs to $H^4(\Omega)$ and is $C^2$ by the Sobolev inequality, see e.g. [E98], p. 270. □

By Lemma 4.6 the argument of Lemma 4.7 applies to all critical points of $H$ in $P = \lambda$, and can be extended to show arbitrarily high regularity.

**Lemma 4.8** *Let $u_* \in H^1(\mathbb{R}^2)$ satisfy $H(u_*) = I_\lambda$, $\lambda > \lambda_0$. Then $u_*$ can be chosen to be a real, positive function.*

*Proof.* Let $U$ be the set where $u_* \neq 0$. In $U$ we can use the polar representation $u_* = R(x)e^{i\Theta(x)}$, with $R$, $\Theta$ real $C^2$ functions in $U$, and $R$ positive. The quadratic part of $H$ is

$$\int_{\mathbb{R}^2} |\nabla u|^2\, dx = \int_{\mathbb{R}^2} (|\nabla R|^2 + R^2|\nabla\Theta|^2)\, dx, \tag{4.26}$$

while the quartic part of $H$ and $P$ depend only on $R$. If $u_*$ is not real and positive in $U$ we can then strictly decrease $H$ by considering the new function $w_* = e^{-i\Theta}u_*$. This contradicts the assumption that $u_*$ is a minimizer. Thus $u_*$ is nonnegative. By the $C^2$ regularity of the minimizer $u_*$, $\psi = u_*$ satisfies (2.11), that is

$$-\frac{D}{2}\Delta\psi - \omega\psi = 2A\psi G(|\psi|^2) \geq 0, \quad \forall x \in \mathbb{R}^2. \tag{4.27}$$

Since $\psi \geq 0$, $\forall x \in \mathbb{R}^2$, we have that $\psi$ is strictly positive on any open ball in $\mathbb{R}^2$ by Hopf's lemma, see e.g. [E98], p.519. □

**Lemma 4.9** *Let $u_* \in H^1(\mathbb{R}^2)$ be a real positive function satisfying $H(u_*) = I_\lambda$, $\lambda > \lambda_0$. Then $u_*$ is radial and decreasing.*

*Proof.* By Lemmas 4.7, 4.8 the minimizer $f = u_* \in H^1(\mathbb{R}^2)$ can be chosen to be positive and smooth. It is therefore decaying at infinity in the sense of the rearrangement lemmas. The same applis to $f^2$. If $f$ (equivalently $f^2$) is not radial, then by Lemmas 3.4, 3.5 we have $H(\overline{f}) < H(f)$, with $P(\overline{f}) = P(f)$, a contradiction. □

By the above, minimizers of $H$ for $P = \lambda > \lambda_0$ are, up to translation and global phase rotation, of the form $f(|x|)$, with $f(|x|)$ a positive, $C^2$ solution of (2.11). By $\theta = G(|f|^2)$, and the properties of the kernel $K_0$, we check that $\theta$ is also $C^2$, positive, and radial.

To study these minimizers in more detail we look for radial solutions $u(r)$, $\theta(r)$, $r = |x|$, of (2.11)

$$u'' + r^{-1}u' \;=\; -\frac{2\omega}{D}u - \frac{4A}{D}\theta u, \tag{4.28}$$

$$\theta'' + r^{-1}\theta' \;=\; m^2\theta - \frac{A}{\nu}u^2. \tag{4.29}$$

Since the $u''(0)$, $\theta''(0)$ are well defined we will require that $u'(0) = \theta'(0) = 0$. These equations are studied numerically in Section 6.

Also, by (2.11), and the fact that $I_\lambda < 0$ for $\lambda > \lambda_0$, we have

$$\omega = \lambda^{-1}(I_\lambda - V(u)) < 0, \tag{4.30}$$

where $V$ is the quartic part of $H$.

# 5 Threshold bounds and decay of small solutions

We now prove Theorem 2.13.

*Proof.* To estimate the power $\lambda_0$ we consider again the form of the Hamiltonian (4.13), written as

$$H = \lambda(\mathcal{A} - \mathcal{B}\lambda), \quad \mathcal{A} = \frac{D}{2}\frac{I_{22}}{4s^2 I_2}, \quad \mathcal{B} = \frac{A}{2\pi}\frac{I_4(s)}{I_2^2}, \tag{5.1}$$

and we have $H(v) < 0$ provided that

$$\lambda > \frac{\mathcal{A}}{\mathcal{B}} = \frac{\pi D}{2A}\frac{I_2 I_{22}}{s^2 I_4(s)}. \tag{5.2}$$

We show that the ratio $\mathcal{A}/\mathcal{B}$ can be bounded above by a quantity that is independent of $s$ and $f$.

We may assume that $f \in C_0^\infty(\mathbb{R}^2)$. Also, by (4.10), (4.11), (4.12), we have

$$I_2 = (2\pi)^{-1}||f||^2_{L^2(\mathbb{R}^2)}, \quad I_{22} = (2\pi)^{-1}||\nabla f||^2_{L^2(\mathbb{R}^2)}, \quad I_4(s) = (2\pi)^{-1}\int_{\mathbb{R}^2} f^2\ (K_{0,sm} * f^2), \tag{5.3}$$

with $K_{0,\mu}(r)$ as in (2.8).

We will give two bounds based on different estimates of the quartic part of the energy $I_4(s)$.

In the first argument we set $s = 1$, suitably redefining $f$. We use the fact that $G$ is bounded in $L^2$, with operator norm $\mathcal{C}_G$, and the Gagliardo-Nirenberg inequality. Note also that by (2.4), $\mathcal{C}_G \leq A\nu^{-1}m^{-2}$. Then by (5.3)

$$\begin{aligned} I_4(s) &\leq \nu(2\pi A)^{-1}\left(\int_{\mathbb{R}^2}|G(|f|^2|^2\right)^{1/2}\left(\int_{\mathbb{R}^2}|f|^4\right)^{1/2} \\ &\leq \nu(2\pi A)^{-1}\mathcal{C}_G\int_{\mathbb{R}^2}|f|^4 \\ &\leq (2\pi m^2)^{-1}(\kappa_{2,4,\frac{1}{2}})^4||f||^2_{L^2(\mathbb{R}^2)}||\nabla f||^2_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{5.4}$$

Therefore

$$\frac{\mathcal{A}}{\mathcal{B}} \geq \frac{D\nu}{2A^2}\frac{m^2}{(\kappa_{2,4,\frac{1}{2}})^4}, \tag{5.5}$$

which we denote $\lambda_1$.

Alternatively, letting $\tau = s^{-1}$,

$$s^2 I_4(s) = (2\pi)^{-1}\int_{\mathbb{R}^2} f^2\ (\tau^{-2}K_{0,\tau^{-1}m} * f^2), \tag{5.6}$$

and using Lemma 3.11

$$\tau^{-2}(K_{0,\tau^{-1}m} * f^2)(x) \leq \sup_{\tau>0}\int_{\mathbb{R}^2}\tau^{-2}K_{0,m}(\tau^{-1}(x-y))f^2(y)dy \leq ||K_{0,m}||_{L^1(\mathbb{R}^2)}\mathcal{M}(f^2)(x). \tag{5.7}$$

Applying (5.7) to (5.6), and using Hölder we have

$$\begin{aligned} s^2 I_4(s) &\leq (2\pi)^{-1} ||K_{0,m}||_{L^1(\mathbb{R}^2)} \int_{\mathbb{R}^2} f^2(x)\, \mathcal{M}f^2(x)\, dx \\ &\leq (2\pi)^{-1} ||K_{0,m}||_{L^1(\mathbb{R}^2)} \left( \int_{\mathbb{R}^2} (f^2(x))^2\, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} (\mathcal{M}(f^2)(x))^2\, dx \right)^{\frac{1}{2}}, \end{aligned} \tag{5.8}$$

$\forall s > 0$. The integral of the maximal function is estimated using the Hardy-Littlewood inequality of Lemma 3.10 as

$$\int_{\mathbb{R}^2} (\mathcal{M}(f^2)(x))^2\, dx \leq c_{2,2} \int_{\mathbb{R}^2} (f^2(x))^2\, dx. \tag{5.9}$$

Then (5.8) becomes

$$s^2 I_4(s) \leq (2\pi)^{-1} (c_{2,2})^{\frac{1}{2}} ||K_{0,m}||_{L^1(\mathbb{R}^2)} \int_{\mathbb{R}^2} f^4, \quad \forall s > 0. \tag{5.10}$$

The last integral is estimated by the Gagliardo-Nirenberg inequality, so that

$$s^2 I_4(s) \leq (2\pi m^2)^{-1} (c_{2,2})^{\frac{1}{2}} (\kappa_{2,4,\frac{1}{2}})^4 ||K_0||_{L^1(\mathbb{R}^2)} ||f||^2_{L^2(\mathbb{R}^2)} ||\nabla f||^2_{L^2(\mathbb{R}^2)}, \quad \forall s > 0. \tag{5.11}$$

We have here also used $||K_{0,m}||_{L^1(\mathbb{R}^2)} = m^{-2} ||K_{0,m}||_{L^1(\mathbb{R}^2)}$, by (2.8). Therefore, by (5.11), (5.1), and (5.3) the powers of $||f||_{L^2(\mathbb{R}^2)}$, $||\nabla f||_{L^2(\mathbb{R}^2)}$ are the same in the numerator and denominator, thus

$$\frac{\mathcal{A}}{\mathcal{B}} \geq \frac{D\nu}{2A^2} \frac{m^2}{(c_{2,2})^{\frac{1}{2}} (\kappa_{2,4,\frac{1}{2}})^4 ||K_0||_{L^1(\mathbb{R}^2)}}, \tag{5.12}$$

denoted $\lambda_2$. □

**Remark 5.1** *The use of radial functions in bounding the ratio $\frac{\mathcal{A}}{\mathcal{B}}$ is not essential. By Lemmas 3.4, 3.5 we can always use radial rearrangements to decrease $\mathcal{A}$, and increase $\mathcal{B}$.*

In what follows we show the decay of small solutions of the initial value problem, as in Theorem 2.6. The argument is based on Picard iteration for $u = \mathcal{G}(u)$, see (4.1) in the proof of Theorem 2.1, in $L^4([0,T], L^4(\mathbb{R}^2))$. The argument we use is essentially that of [CW89] for the cubic NLS in $\mathbb{R}^2$. We complete the proof of Theorem 2.6 by showing that for small initial conditions in $H^1$ these rougher solutions coincide with the continuous trajectories of Theorem 2.1.

Let $\delta > 0$, and define $U_{2\delta}([0,T])$ by

$$U_{2\delta}([0,T]) = \{ v \in L^4([0,T], L^4(\mathbb{R}^2)) : \quad ||v||_{L^4([0,T], L^4(\mathbb{R}^2))} \leq 2\delta \}. \tag{5.13}$$

Note that $(q,r) = (4,4)$ is an admissible pair for the Strichartz estimates. Although the factor $D/2$ can be scaled away we prefer to use the rescaled free Schrödinger propagator $\tilde{S}(t) = S(\frac{D}{2}t)$, see Remark 3.9.

Let $U_{2\delta} = U_{2\delta}([0,\infty))$, and $L^{4,4} = L^4([0,\infty), L^4(\mathbb{R}^2))$.

The following implies the existence of a unique solution to the initial value problem $u = \mathcal{G}(u)$ in $L^{4,4}$ (and $L^4([0,T], L^4(\mathbb{R}^2))$ for arbitrary $T \geq 0$).

**Lemma 5.2** *Let $\delta \in (0, \delta_0)$. There exists $\tilde{\delta}$ (depending on $\delta$) such that if $||\phi||_{L^2(\mathbb{R}^2)} < \tilde{\delta}$ then, $\mathcal{G}$ is a contraction in $U_{2\delta}$.*

*Proof.* We need to show that (i) $\mathcal{G}$ maps $U_{2\delta}$ to its interior, and (ii) that $\mathcal{G}$ is a contraction in $U_{2\delta}$.

First, by the first Strichartz estimate (3.12), given $\delta > 0$, we can choose $||\phi||_{L^2(\mathbb{R}^2)}$ small enough so that

$$||\tilde{S}(\cdot)\phi||_{L^{4,4}} < \delta. \tag{5.14}$$

To show that $\mathcal{G}$ maps $U_{2\delta}$ to its interior it suffices to show that $||F(u)(\cdot)||_{L^{4,4}} < \delta$.

Let $\sigma = 4$, $\sigma' = 4/3$. By Hölder, and the boundedness of $G$ in $L^2$ we have

$$\begin{aligned} \int_{\mathbb{R}^2} |\theta u|^{\sigma'} &\leq \left( \int_{\mathbb{R}^2} |\theta|^{\frac{3}{2}\sigma'} \right)^{1/r} \left( \int_{\mathbb{R}^2} |u|^{3\sigma'} \right)^{1/p} \\ &\leq \left( \int_{\mathbb{R}^2} |\theta|^2 \right)^{2/3} \left( \int_{\mathbb{R}^2} |u|^4 \right)^{1/3} \leq \mathcal{C}_G^{4/3} \int_{\mathbb{R}^2} |u|^4, \end{aligned} \tag{5.15}$$

therefore

$$\left( \int_0^\infty \int_{\mathbb{R}^2} |\theta u|^{\sigma'} \right)^{1/\sigma'} \leq \left( \int_0^\infty \mathcal{C}_G^{4/3} \int_{\mathbb{R}^2} |u|^4 \right)^{3/4} \leq \mathcal{C}_G(||u||_{L^{4,4}})^3. \tag{5.16}$$

Applying the second Strichartz estimate (3.14) to $F$ in (4.2), and using (5.16)

$$||F||_{L^{4,4}} \leq \tilde{C}_2 ||\theta u||_{L^{\sigma',\sigma'}} \leq \tilde{C}_2 \mathcal{C}_G(||u||_{L^{4,4}})^3. \tag{5.17}$$

Thus, $u \in U_{2\delta}$, $\delta$ sufficiently small, imply

$$||F||_{L^{4,4}} \leq C_2 \mathcal{C}_G (2\delta)^3 < \delta, \tag{5.18}$$

as required.

To see that $\mathcal{G}$ is a contraction we use the linearity of $G$ to write

$$\begin{aligned} G(|u|^2)u - G(|v|^2)v &= G(|u|^2)(u - v) + G(|u|^2 - |v|^2)v \\ &= G(|u|^2)(u - v) + G((|u| + |v|)(|u| - |v|))v. \end{aligned} \tag{5.19}$$

By the positivity of the kernel $K_0$ we also have that $h \leq f$ everywhere implies $G(h) \leq G(f)$ everywhere, thus (5.19) becomes

$$|G(|u|^2)u - G(|v|^2)v| \leq G(|u|^2 + |v|^2)|u - v| + G((|u| + |v|)|u - v|))(|u| + |v|). \tag{5.20}$$

By the definition of $\mathcal{G}$, and the second Strichartz estimate (3.14) applied to $F$ in (4.2),

$$\begin{aligned} ||\mathcal{G}(u) - \mathcal{G}(v)||_{L^{4,4}} &= ||F(u) - F(v)||_{L^{4,4}} \\ &\leq \tilde{C}_2 ||G(|u|^2)u - G(|v|^2)v||_{L^{\sigma',\sigma'}} \\ &= \tilde{C}_2 \left( \int_0^\infty \int_{\mathbb{R}^2} |G(|u|^2)u - G(|v|^2)v|^{\sigma'} \right)^{1/\sigma'}, \end{aligned} \tag{5.21}$$

with $\sigma' = 4/3$. Using (5.20) and the triangle inequality for the norm in $L^{\sigma',\sigma'}$ we then have

$$||\mathcal{G}(u) - \mathcal{G}(v)||_{L^{4,4}} \le ||G(|u|^2 + |v|^2)|u - v|||_{L^{\sigma',\sigma'}} + ||G((|u| + |v|)|u - v|))(|u| + |v|)||_{L^{\sigma',\sigma'}}. \quad (5.22)$$

Using Hölder repeatedly, and the boundedness of $G$ in $L^2$, the first term in (5.22) is estimated as

$$\begin{aligned}
||G(|u|^2 + |v|^2)|u - v|||_{L^{\sigma',\sigma'}} &\le \left[\int_0^\infty \left(\int_{\mathbb{R}^2} |G(|u|^2 + |v|^2)|^{\frac{3}{2}\sigma'}\right)^{2/3} \left(\int_{\mathbb{R}^2} |u - v|^{3\sigma'}\right)^{1/3}\right]^{1/\sigma'} \\
&\le \left[\int_0^\infty \mathcal{C}_G^{4/3} \left(\int_{\mathbb{R}^2} (|u|^2 + |v|^2)^2\right)^{2/3} \left(\int_{\mathbb{R}^2} |u - v|^4\right)^{1/3}\right]^{3/4} \\
&\le \mathcal{C}_G \left[\left(\int_0^\infty \int_{\mathbb{R}^2} (|u|^2 + |v|^2)^2\right)^{2/3} \left(\int_0^\infty \int_{\mathbb{R}^2} |u - v|^4\right)^{1/3}\right]^{3/4} \\
&\le \mathcal{C}_G \left(\int_0^\infty \int_{\mathbb{R}^2} ||u|^2 + |v|^2|^2\right)^{1/2} ||u - v||_{L^{4,4}} \\
&\le \sqrt{2}\mathcal{C}_G \left(||u||^2_{L^{4,4}} + ||v||^2_{L^{4,4}}\right) ||u - v||_{L^{4,4}}. \qquad (5.23)
\end{aligned}$$

Letting $\tilde{w} = (|u| + |v|)|u - v|$ we similarly estimate the second term in (5.22) as

$$\begin{aligned}
||G((\tilde{w})(|u| + |v|)||_{L^{\sigma',\sigma'}} &\le \left[\int_0^\infty \left(\int_{\mathbb{R}^2} |G(\tilde{w})|^{\frac{3}{2}\sigma'}\right)^{2/3} \left(\int_{\mathbb{R}^2} (|u| + |v|)^{3\sigma'}\right)^{1/3}\right]^{1/\sigma'} \\
&\le \left[\int_0^\infty \mathcal{C}_G^{4/3} \left(\int_{\mathbb{R}^2} |\tilde{w}|^2\right)^{2/3} \left(\int_{\mathbb{R}^2} (|u| + |v|)^4\right)^{1/3}\right]^{3/4} \\
&\le \mathcal{C}_G \left[\int_0^\infty \left(\int_{\mathbb{R}^2} (|u| + |v|)^4\right)^{2/3} \left(\int_{\mathbb{R}^2} |u - v|^4\right)^{1/3}\right]^{3/4} \\
&\le \mathcal{C}_G \left[\left(\int_0^\infty \int_{\mathbb{R}^2} (|u| + |v|)^4\right)^{2/3} \left(\int_0^\infty \int_{\mathbb{R}^2} |u - v|^4\right)^{1/3}\right]^{3/4} \\
&\le \mathcal{C}_G \left(\int_0^\infty \int_{\mathbb{R}^2} (|u| + |v|)^4\right)^{1/2} ||u - v||_{L^{4,4}} \\
&\le \sqrt{8}\mathcal{C}_G \left(||u||^2_{L^{4,4}} + ||v||^2_{L^{4,4}}\right) ||u - v||_{L^{4,4}}. \qquad (5.24)
\end{aligned}$$

By (5.21), (5.23), (5.24), and $u \in U_{2\delta}$ we have

$$||\mathcal{G}(u) - \mathcal{G}(v)||_{L^{4,4}} \le 8\mathcal{C}_G\tilde{C}_2(2\delta)^2||u - v||_{L^{4,4}} \qquad (5.25)$$

and thus for $\delta$ sufficiently small $\mathcal{G}$ is a contraction in $U_{2\delta}$. □

By (5.18), (5.25) we can choose $\delta \in (0, \delta_0)$ with $32\tilde{C}_2\mathcal{C}_G\delta_0^2 < 1$. Then by (5.14) we can choose $\tilde{\delta}$ such that $\tilde{C}_1\tilde{\delta} < \delta$.

We now complete the proof of Theorem 2.6.

*Proof.* We consider an initial condition $u(0) \in H^1(\mathbb{R}^2)$ and the corresponding unique solution $u$ of Theorem (2.1). We also assume that $\phi = u(0)$ also satisfies the small $L^2-$norm condition (5.14), with $\delta$ as in Lemma 5.2.

It enough to show that this trajectory, restricted to any time interval $[0, T]$, belongs to the interior of $U_{2\delta}([0, T])$. It thus coincides with the solution of Lemma 5.2, i.e. the unique fixed point of $\mathcal{G}$ in $U_{2\delta}([0, T])$, and belongs to $L^4([0, T], L^4(\mathbb{R}^2))$, $\forall T > 0$.

We note that the conservation of the $L^2-$norm, the Gagliardo-Nirenberg inequality, and the fact that the $H^1-$norm of $u$ remains bounded for all times, imply that $u$ also belongs to $C^0([0, \infty), L^4(\mathbb{R}^2))$, moreover its $L^4-$norm remains bounded for all times. Let $q(t)$ be the $L^4([0, t], L^{\mathbb{R}^2})$ norm of $u$ restricted to $[0, t]$. Clearly, $q$ is a continuous, non-decreasing function of $t \geq 0$, with $q(0) = 0$. We therefore have that for $t > 0$ small enough $u \in U_{2\delta}([0, t])$. Since $u$ is the fixed point of $\mathcal{G}$, it also belongs to the interior of $U_{2\delta}([0, t])$ by the smallness assumption on the $L^2-$norm of $\phi$. Suppose that there exists a time $t_1 > 0$ with $q(t_1) > 2\delta$. Then by continuity, there exists $\tilde{t} \in (0, t_1)$ with $q(\tilde{t}) = 2\delta$. But $u$ restricted to $[0, \tilde{t}]$ is a fixed point of $\mathcal{G}$ and belongs to the boundary of $U_{2\delta}([0, \tilde{t}])$, a contradiction, since the condition on the $L^2-$norm of $\phi$ implies that $\mathcal{G}$ maps $U_{2\delta}([0, \tilde{t}])$ to its interior. Thus the trajectory $u$ of Theorem 2.1 remains in the interior of $U_{2\delta}([0, T])$, $\forall T > 0$.

We therefore have Theorem 2.6, with $\lambda_3 = \tilde{\delta}$. □

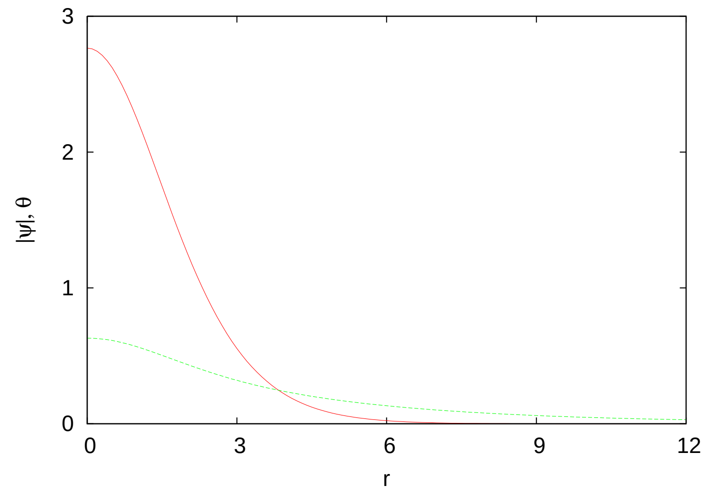
# 6 Numerical solutions and small solutions in bounded domains

We now examine numerical solutions of the radial equation (2.11). We are interested in positive, decaying solutions. Such solutions can be found (approximated) numerically, but we also see some departures from the infinite plane theory above that we attribute to differences between the infinite and finite domain soliton solutions. We write (2.1) and (2.2) with $u(r, t) = \psi(r)e^{-i\omega t}$ (which leads to (2.11)) and obtain

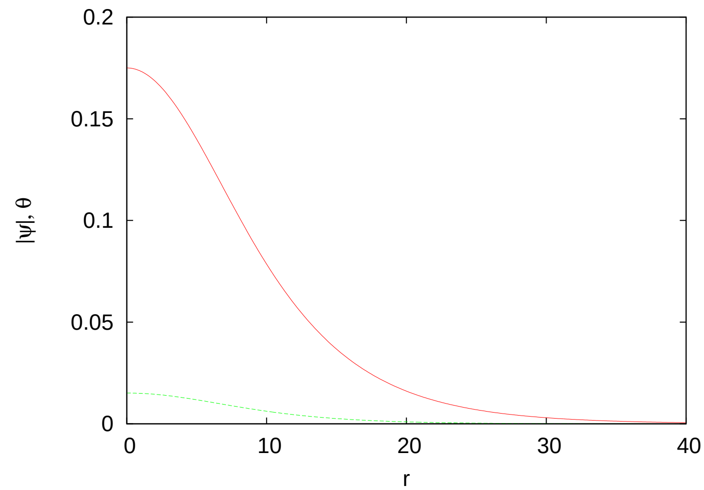
$$L\psi + \omega\psi = 0, \quad \nu\Delta\psi - q\theta = -\psi^2, \quad \text{where} \quad L = \frac{1}{2}\psi'' + \frac{1}{2}r^{-1}\psi' + 2\theta\psi, \tag{6.1}$$
$$\omega = -\frac{(L\psi, \psi)}{(\psi, \psi)}, \quad (f, g) = \int_0^\infty fg dr,$$

where $A = D = 1$ and $q = m^2\nu$. Furthermore, (6.1) is discretized in a finite computational domain with $r \in [0, R_{max}]$. We impose Dirichlet boundary conditions $\psi(R_{max}) = \theta(R_{max}) = 0$. We typically use $R_{max} = 100$. In the range of parameters we examined, $\psi$ and $\theta$ have decayed to $O(10^{-10})$ at $r = 100$, and increasing the computational domain does not cause the solutions to change. The spatial discretization uses second-order central finite differences over a uniform grid with spacings $\Delta r = 0.1$. Decreasing the value of $\Delta r$ does not alter the results, so numerically converged results are obtained. The integral $(f, g)$ is similarly evaluated by quadrature over the computational domain.

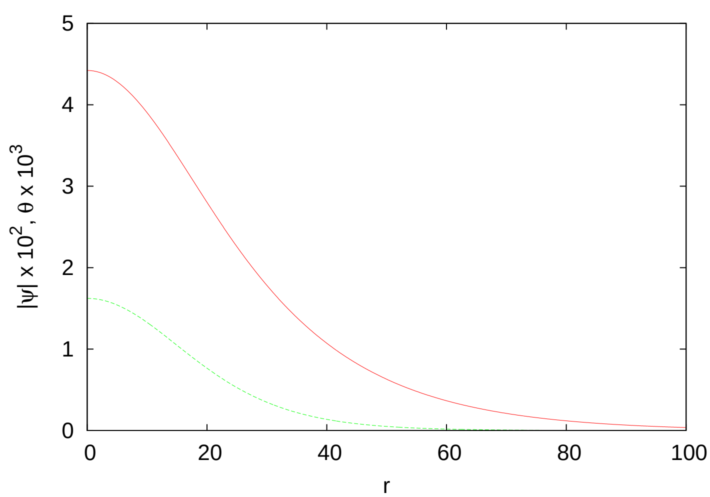
The imaginary time iterative method of [LMS13] is used to find numerically solitary wave profiles. This numerical method uses a given optical power $P = (\psi^2, r)$, to converge to a member of the solitary wave family. The corresponding propagation constant $\omega$ is found by solving the third of (6.1).




(a)




(b)




(c)



Figure 1: (color online) The nematic solitary wave, $|\psi|$ and $\theta$ versus $r$. Shown are $|\psi|$ (red, solid lines) and $\theta$ (green, dashed lines). The parameters are (a) $P = 10$, (b) $P = 1$ and (c) $P = 0.5$ with the other parameters $\nu = 25$ and $q = 1$. (In (c) we used $R_{max} = 200$.)

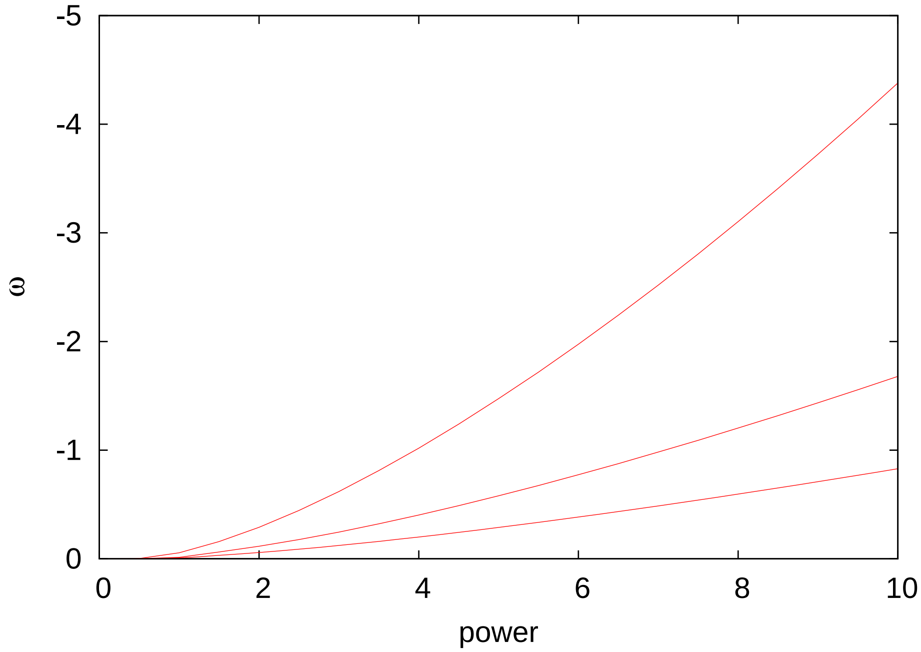



Figure 2: (color online) The propagation constant versus power, $\omega$ versus $P$, for the nematic solitary wave. Shown are curves for $\nu = 25$ (upper curve), $\nu = 12.5$ and $\nu = 5$ (lower curve) (all red solid lines). The other parameter $q = 1$.

Figure 1(a), (b) and (c) show nematic solitary waves, $|\psi|$ and $\theta$, versus $r$. Shown are the electric field amplitude $|\psi|$ and the director response $\theta$. The parameters are (a) $P = 10$, (b) $P = 1$ and $P = 0.5$ with $\nu = 25$ and $q = 1$. For $P = 10$ the propagation constant is $\omega = -0.828$. This wave has a large amplitude, with $|\psi| = 2.77$ and is relatively more localized, with a half-width (the location at which the amplitude is half the peak amplitude) of $w = 1.85$. As $\nu$ is large, the director response, which has a peak amplitude of $\theta = 0.630$, is broader than the electric field, indicating the nonlocal nature of the solitary wave.

As the power is reduced the amplitude of the nematic wave decreases and its width increases. For $P = 1$ the propagation constant is $\omega = -1.55 \times 10^{-2}$, the peak amplitude is $|\psi| = 0.175$ and the half-width $w = 9.2$. The director response $\theta$ now has a very small magnitude (as $\theta = O(|\psi|^2)$), and has a peak amplitude of $\theta = 1.52 \times 10^{-2}$. For $P = 0.5$ the propagation constant is $\omega = -7.14 \times 10^{-4}$ and the peak amplitude is decreased and the width increased further. The peak amplitude is $|\psi| = 4.42 \times 10^{-2}$ and the half-width $w = 25.4$ has a peak amplitude of $\theta = 1.62 \times 10^{-3}$.

Figure 2 shows the propagation constant versus power, $\omega$ versus $P$, for the nematic solitary wave. The parameters are $\nu = 25$, $\nu = 12.5$ and $\nu = 5$, all with $q = 1$. The figure shows that there is a unique correspondence between power and propagation constant. We have not examined, however, the possibility of bifurcations leading to other branches connected to the calculated one. We also see that, for a given power level, the magnitude of the propagation constant increases as the nonlocality parameter $\nu$ increases.

The numerical results above show examples of radial, positive and decreasing soliton solutions,

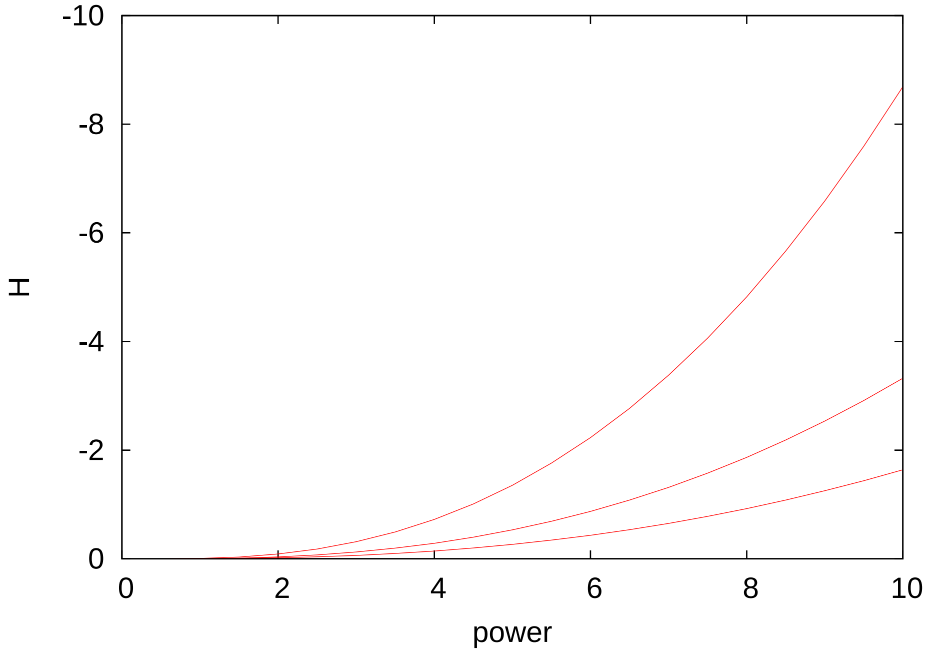



Figure 3: (color online) The Hamiltonian versus power, $H$ versus $P$, for the nematic solitary wave. Shown are curves for $\nu = 25$ (lower curve), $\nu = 12.5$ and $\nu = 5$ (upper curve) (all red solid lines). The other parameter $q = 1$.

as expected by the theory in the plane. On the other hand, the numerical study does not see any minimum power level for nematic solitary waves. We argue below that this should follow from the theory of (6.1) in the finite domain (disc).

We further use the numerical solutions to compute the Hamiltonian

$$H = \frac{1}{2}(\psi_r^2, r) - (r\theta, \psi), \tag{6.2}$$

and the ratio

$$\mathcal{R} = \frac{P\langle\psi_r^2, r\rangle}{\langle r, \theta\psi\rangle}, \tag{6.3}$$

i.e. $\mathcal{R} = \mathcal{A}/2\mathcal{B}$, see (5.1). The ratio is computed using the finite domain functions and integrals, and $\theta$ is the solution of the second equation in (6.1) with Dirichlet boundary conditions. Clearly, a function $f$ defined in $[0, R_{max}]$ can be extended to function $\tilde{f}$ in $[0, \infty)$ as $\tilde{f}(r) = 0$, $r > R_{max}$, note however that the computed $\theta$ is slightly different from $G(\tilde{f}^2)$. Since $u$ decays rapidly we expect that this discrepancy is small.

Figure 3 shows the Hamiltonian versus power, $H$ versus $P$. The parameters are $\nu = 25$, $\nu = 12.5$ and $\nu = 5$, all with $q = 1$. The curves suggest that $H$ is a negative quantity for for sufficiently large values of the power. In fact at small values of $P \leq 0.495$, $H$ becomes positive. The magnitude of the Hamiltonian for $P < 0.495$ does not exceed $\mathrm{O}(10^{-5})$. Also, for a fixed power $H$ increases as the nonlocality parameter $\nu$ increases.

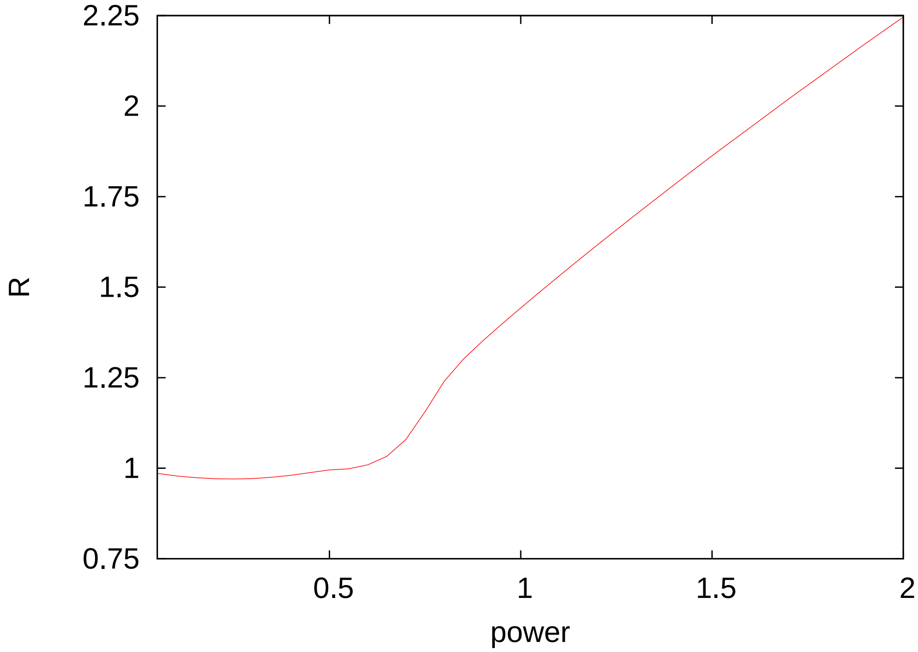



Figure 4: (color online) The ratio (6.3) versus power, $\mathcal{R}$ versus $P$, for the nematic solitary wave. The parameters are $\nu = 5$ and $q = 1$ (red solid line).

Figure 4 shows the ratio (6.3) versus power, $\mathcal{R}$ versus $P$. The parameters are $\nu = 5$ and $q = 1$. The curve shows that $\mathcal{R}$ is a positive quantity for all values of the power. As $P \to 0$ the ratio approaches a value $\mathcal{R} \approx 0.99$. For large enough power, the ratio $\mathcal{R}$ increases with $P$. Hence this quantity does not approach zero as the power $P \to 0$ but asymptotes towards a finite limit. Varying the value of $\nu$ does not change the qualitative features of Figure 4. The ratio $\mathcal{R} \approx 1$ in the limit of small power and increases in value for large power, for all values of $\nu$. As both the numerator and denominator of (6.3) approach zero as $P \to 0$ the calculation of (6.3) breaks down in this limit, due to round off error. Our calculation is done for a minimum $P = 0.05$ at which $\mathcal{R} = 0.985$ and the numerator and denominator are both $O(10^{-6})$.

The above calculation is consistent with the existence of a lower bound on $\mathcal{A}/\mathcal{B}$, as shown in Section 5. Also, the change of sign of the energy, and the existence of positive energy states are consistent with the analogue of expression $H = \lambda(\mathcal{A} - \mathcal{B}\lambda)$, $\lambda = P$, of (5.1) for the finite domain: as $P$ decreases and $\mathcal{A}/\mathcal{B}$ remains bounded above, $H$ becomes positive. As $P$ is decreased further $H$ remains small, and vanishes since it is proportional to $\lambda\mathcal{A}$.

The fact that $P$ vanishes with $\mathcal{A}/\mathcal{B}$ bounded above also implies that ratio of the quadratic part to the quartic part of $H$ increases without bound as the power $P$ vanishes, i.e. the linear part becomes dominant.

A related heuristic for solutions with $P = \epsilon$, $\epsilon \to 0$, is obtained by the rescaling $\psi = \sqrt{\epsilon} v$. Then, $v(r)$ is normalized, $P(v) = 1$, and satisfies

$$-\Delta v = \omega v + 4\epsilon \tilde{G}(v^2) v, \quad v(R_{max}) = 0, \tag{6.4}$$

where $\tilde{G}(v^2) = \theta$, the radial solution of $-\nu\Delta\theta + q\theta = v^2$, with $v(R_{max}) = 0$.

The existence of numerical solutions of arbitrarily small power can be explained by the theory of the radial equation (6.4) in a finite disc with Dirichlet boundary conditions. One approach is through local bifurcation theory, where we can show the existence of small nontrivial solutions with $\omega$ near the eigenvalues of the radial Dirichlet Laplacian.

In particular, consider the abstract nonlinear eigenvalue equation

$$\Gamma\mathbf{x} = (\chi_0 + \chi)\mathbf{x} + g(\mathbf{x}), \tag{6.5}$$

with $\mathbf{x} \in E$, $\Gamma \in L(B, E)$, $B$, $E$ real Banach spaces, and $g : B \to E$ a $C^1$ function in $B$ satisfying $g(0) = 0$, $Dg(0) = 0$, $\chi_0$ an eigenvalue of $\Gamma$ of finite multiplicity. Using e.g. statement II.2,3 of [B72], if $\chi_0$ has odd multiplicity we then have a nontrival solution $(\mathbf{x}, \chi)$ near the trivial solution $\mathbf{x} = 0$, $\chi = 0$.

In the present problem (6.1) can be written as (6.5) by choosing $B$ the closure of $H^2_r(\Omega, \mathbb{R}) \cap H^1_{0,r}(\Omega, \mathbb{R})$ in the $H^2-$norm, $E = L^2(\Omega, \mathbb{R})$, where $\Omega = \{r \in \mathbb{R}^2 : |r| < R_{max}\}$, and the subscript $r$ denotes restriction to radial functions. Also $\Gamma$ is the radial Dirichlet Laplacian, $\chi_0$ its lowest eigenvalue and $g$ the nonlinear term $\psi\theta$ (times a suitable constant) .

The properties of $\Gamma$, $g$ required to apply the abstract result are easy to check, using e.g. the boundedness of $\psi^2 \mapsto \theta(\psi^2)$ in $L^2(\Omega, \mathbb{R})$, and $L^4(\Omega, \mathbb{R}) \subset H^1(\Omega, \mathbb{R})$, and we also use the fact that $\chi_0$ is simple. Thus the existence of a nontrivial solution of arbitrarily small power is guaranteed, and this shows that there is no contradiction with the infinite domain decay result. The bifurcation approach to (6.1) can yield more information, but will not be pursued further here.

The abstract setup of (6.5) also applies to the finite dimensional discretizations used in the numerical implementation.

We also remark that a comparable discrepancy between the finite and infinite problem should occur in discrete NLS equations. The theoretical works of Weinstein [W99], and Stefanov and Kevrekidis [SK05] show that discrete NLS equations with power nonlinearities in infinite lattices can also exhibit critical power/dimension combinations for the existence of analogous soliton-type solutions. In particular, what we see here is analogous to what is seen for the cubic NLS in a two-dimensional infinite lattice. That problem can be considered in a finite sublattice with different (analogues of) boundary conditions, e.g. periodic. The finite version can also preserve the variational structure of the analogue of (2.11), so that the existence of solutions of the form $e^{-i\omega t}\psi_n$, with $n$ the lattice site index, follows from the fact that we are extremizing a smooth function, the analogue of $H$, over a compact set, the $(2N-1)-$sphere $P = \lambda > 0$ in $\mathbb{R}^{2N}$, where $N$ the number of sites in the finite sublattice. Clearly, a minimizer will exist for any $\lambda > 0$, i.e. for arbitrarily small power.

In [FKM97], [F96] the existence of power thresholds for discrete NLS equations in infinite lattices is inferred from secondary bifurcations along the branch bifurcating from the trivial solution at the lowest linear eigenvalue of the discrete Laplacian, specifically the limit of the value of the power at this secondary bifurcation as the lattice size diverges. A similar approach may be applicable to the present case and can lead to a numerical estimate of $\lambda_*$, however a search of bifurcations along the branch exhibited here is left for future work.

## 7 Discussion

We have obtained results on the existence, regularity and symmetry of energy minimizing soliton solutions of the nematicon equation on the plane. We have also provided power threshold bounds for the existence of such solutions, and have shown that any solutions with small power must eventually decay.

For powers above a certain threshold, the Hartree-type nonlocality of the nematicon equation avoids the finite-time blow-up behavior of the two dimensional cubic NLS. For smaller power the two models show qualitatively similar behavior, in the sense that they both lead to decay. A similar situation arises in the two dimensional cubic NLS on a 2-D infinite lattice, as seen in [W99], [SK05].

A more careful comparison of the upper bounds on the power of the negative energy states, and the condition for decay of solutions is left for future work. Also of interest is the structure of the set of constrained energy minimizers $\mathcal{M}_\lambda$, e.g. whether the constrained minimizers are isolated or unique modulo translations and phase rotations. These properties would imply a more precise stability statement.

The numerical study of solitons uses a finite domain with Dirichlet boundary conditions. We find solutions with the properties expected by the infinite plane theory, but we also see solutions of arbitrarily small power. The small power solutions can be understood as branches bifurcating from the eigenfunctions of the Dirichlet Laplacian on the disc. We speculate that the branch of the numerical solutions we find contains a part that corresponds to approximations of solutions to the infinite problem, as well as a part of "spurious" solutions of small power. Following the approach of [FKM97], it is also possible that the computed branch has bifurcation points that separate the two parts, and yield further branches we have not seen. An interesting related problem is the study of threshold and bifurcation phenomena for discretizations or radial equations of NLS type, in finite and (semi-)infinite lattices.

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