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Stability of traveling water waves on the sphere

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Abstract

We study the stability of a class of traveling waves in a model of weakly nonlinear water waves on the sphere. The model describes free surface potential flow of a fluid layer surrounding a gravitating sphere, and the evolution equations are Hamiltonian. For small amplitude oscillations the Hamiltonian can be expanded in powers of the wave amplitude, yielding simpler model equations. We integrate numerically Galerkin truncations of such a model, focusing on a class of traveling and standing waves that are "near-monochromatic" in space, i.e. have amplitude consisting of one spherical harmonic plus small corrections. We observe that such motions are stable for long times. To explain the observed behavior we use methods of Hamiltonian dynamics, first showing that decay to all but a small number of modes must be very slow. To understand the interaction between these modes we obtain general conditions for the long time nonlinear stability of a certain class of periodic orbits in Hamiltonian systems of resonantly coupled harmonic oscillators. © 2001 Published by Elsevier Science B.V. on behalf of IMACS.

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1. Introduction

In recent years there has been considerable effort in applying ideas from dynamical systems to Hamiltonian nonlinear wave equations, and in this contribution we report on some related work on small amplitude gravity water waves in spherical geometry. The system is Hamiltonian and the evolution equations for the spherical harmonic coefficients (modes) of the relevant dependent variables have the form of an infinite system of coupled harmonic oscillators. This structure is preserved under a variety of simplifications to model water wave equations (e.g. shallow water limits, truncations in the wave amplitude, Galerkin projections etc.), so that we can apply Hamiltonian methods to systems of varying complexity.

A first step in our study is a Birkhoff normal form calculation where we eliminate cubic and nonresonant quartic terms in the Hamiltonian by formal canonical transformations. We observe that the quartic normal form Hamiltonian (i.e. with higher order terms omitted) possesses families of periodic orbits that can be interpreted as traveling and standing waves. The amplitude of these waves consists of one spherical

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harmonic plus small corrections and we refer to them as near-monochromatic. Numerical simulations of near-monochromatic waves in Galerkin approximations of the water wave equations suggest that such motions are stable over long times, and the observations are partially explained by identifying additional constants of motion in the quartic Birkhoff normal form Hamiltonian. These quantities evolve slowly in the original equations, and control the distance from the periodic orbits in all but a few directions. Thus, decay of the near-monochromatic waves is due to the interaction of a finite number of modes.

To understand the effect of these interactions on near-monochromatic waves we study analogous normal mode solutions in resonant Hamiltonian systems of weakly coupled harmonic oscillators, and we present criteria for the asymptotic stability of such motions in polynomial and exponential time scales. It is interesting that these conditions involve the size of "Benjamin–Feir" resonant quartic interactions that are analogous to the terms giving rise to linear instabilities in Stokes waves.

2. Hamiltonian formulation of water waves on the sphere

We consider a fluid layer of thickness h surrounding a gravitating sphere of radius b. Using polar coordinates with r the radius, and ϑ and φ the polar and azimuth angles, respectively, the outer surface of the layer is at $r(\vartheta,\varphi) = \rho + \eta(\vartheta,\varphi)$, where $\rho = b + h$. We assume that the flow inside the layer is potential and denote the hydrodynamic potential by ϕ. Then Euler's equations imply that

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial r} - \frac{1}{r^2}\frac{\partial\phi}{\partial\vartheta}\frac{\partial\eta}{\partial\vartheta} - \frac{1}{r^2\sin^2\vartheta}\frac{\partial\phi}{\partial\varphi}\frac{\partial\eta}{\partial\varphi}, \qquad \frac{\partial\phi}{\partial t} = -\frac{1}{2}|\nabla\phi|^2 + \frac{1}{\rho+\eta}, \tag{2.1}$$

at the free (outer) surface

$$\Delta\phi = 0 \quad \text{inside the layer}, \tag{2.2}$$

and

$$\frac{\partial\phi}{\partial r} = 0 \quad \text{at} \quad r = b. \tag{2.3}$$

Note that at each instant, the wave amplitude $\eta(\vartheta,\varphi)$ and the surface potential $\Phi(\varphi,\vartheta) = \phi(\varphi,\vartheta,\rho + \eta(\vartheta,\varphi))$ determine the potential ϕ inside the layer uniquely.

The equations for free surface potential flow also posses a Hamiltonian structure (see [1]). The canonical variables are the wave amplitude η and the hydrodynamic potential at the surface Φ, the Hamiltonian H is the total energy (kinetic + potential) of the system (see [2]), while (2.1)–(2.3) are equivalent to

$$\frac{\partial\eta}{\partial t} = \left(\frac{\rho+\eta}{\rho}\right)^{-2}\frac{\delta H}{\delta\Phi}, \qquad \frac{\partial\phi}{\partial t} = -\left(\frac{\rho+\eta}{\rho}\right)^{-2}\frac{\delta H}{\delta\eta}, \tag{2.4}$$

with the Hamiltonian H given by

$$H = \frac{1}{2}\int_{S^2}\Phi R\left.\frac{\partial\phi}{\partial\hat{n}}\right|_{r=\rho+\eta(\vartheta,\varphi)}\mathrm{d}A_r - \frac{1}{2}\int_{S^2}\mathrm{d}A_r. \tag{2.5}$$

Here $\hat{n}$ is the outward unit normal at the free surface, $\mathrm{d}A_r = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, and $R\,\mathrm{d}A_r$ is the area element of the free surface. Introducing the Dirichlet–Neumann operator $G(\eta)$ by

$$G(\eta)\Phi = R\frac{\partial\phi}{\partial\hat{n}}\bigg|_{r=\rho+\eta(\vartheta,\varphi)}, \tag{2.6}$$

we can write the Hamiltonian as

$$H = \frac{1}{2}\int_{S^2}\Phi G(\eta)\Phi\,\mathrm{d}A_r - \frac{1}{2}\int_{S^2}\mathrm{d}A_r. \tag{2.7}$$

The Dirichlet–Neumann operator can be exactly expanded in powers of the wave amplitude η, and we may write $H = \sum_{j=0}^{\infty} H_j$, with H_j of order $j+2$ in the canonical variables (see [3,4]). It is convenient to use the variables $\tilde{\eta}$ and $\tilde{\Phi}$, defined by $\tilde{\eta} = \eta(1+\eta/2\rho)$, $\tilde{\Phi} = \Phi(1+\eta/\rho)$. From now on all quantities will be expressed in terms of $\tilde{\eta}$ and $\tilde{\Phi}$ and we drop the tilde from the notation. We also modify the H_j so that they still consist of the terms of order $j+2$ in the new variables.

Expanding η, Φ as $\eta = \sum_\gamma \eta_\gamma Y_\gamma$, $\Phi = \sum_\gamma \Phi_\gamma Y_\gamma$, with $Y_\gamma(\vartheta,\varphi)$, $\gamma = [l,m]$, $l = 1,2,\ldots$, $m = -l,\ldots,l$ the spherical harmonics, the evolution of the modes η_γ, Φ_γ is given by

$$\dot{\eta}_\gamma = \frac{\partial H}{\partial \Phi_\gamma^*}, \qquad \dot{\Phi}_\gamma = -\frac{\partial H}{\partial \eta_\gamma^*}, \tag{2.8}$$

where η_γ^*, Φ_γ^* are the complex conjugates of η_γ, Φ_γ, respectively. The quadratic part of the Hamiltonian is

$$H_0 = \frac{\rho^2}{2}\sum_\gamma\left(\frac{u'_\gamma(\rho)}{u_\gamma(\rho)}\Phi_\gamma\Phi_\gamma^* + \eta_\gamma\eta_\gamma^*\right), \text{ with } u_\gamma(r) = (l+1)\left(\frac{r}{b}\right)^l + l\left(\frac{b}{r}\right)^{l+1}, \tag{2.9}$$

i.e. the dispersion relation is $\omega_\gamma^2 = u'_\gamma(\rho)u_\gamma^{-1}(\rho)$, while the cubic part of H is

$$\begin{aligned} H_1 &= \sum_{\gamma_1,\gamma_2,\gamma_3} I_{\gamma_1,\gamma_2,\gamma_3}\Phi_{\gamma_1}\Phi_{\gamma_2}\eta_{\gamma_3}, \text{ with } I_{\gamma_1,\gamma_2,\gamma_3} \\ &= \frac{\rho^2}{2}\left(\frac{u''_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} - \frac{u'_{\gamma_1}(\rho)}{u_{\gamma_1}(\rho)}\frac{u'_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)}\right)\int_{S^2} Y_{\gamma_1}Y_{\gamma_2}Y_{\gamma_3} + \frac{1}{2}\int_{S^2} Y_{\gamma_1}\nabla Y_{\gamma_2}\cdot\nabla Y_{\gamma_3}. \end{aligned} \tag{2.10}$$

The quartic part of the Hamiltonian is computed explicitly in [2], where we also give a recursive expression for higher order nonlinearities. According to the dimensional analysis of [2] the Hamiltonian can be written as $H = \sum_{j=0}^{\infty}\epsilon^j H_j$ with ϵ the ratio of a typical wave amplitude to the depth h. Moreover, in each H_j, $j > 0$ we can factor out a term β^j, where $\beta = h/b$. Weakly nonlinear waves will correspond to $\epsilon \ll 1$.

3. Birkhoff normal forms and near-monochromatic waves

As a first step in understanding the dynamics of weakly nonlinear water waves on the sphere we study the resonant interactions between the various modes. A systematic way to do this is to calculate Birkhoff normal forms, where we can also take advantage of the Hamiltonian formulation of the problem. In the normal form systems we can identify several interesting exact solutions, and here we will be interested in certain “near-monochromatic” traveling and standing wave solutions in which the wave amplitude consists of a spherical harmonic plus small corrections.

To perform the normal form we can use the spectral form of the equation. Canonical transformations can be computed explicitly as in the finite dimensional case (e.g. [5]). The computations involve formal power series (although convergence can also be shown near the origin in analytic topologies). We find (see [2]) that all cubic terms of the Hamiltonian can be eliminated, in particular we can bound away from the cubic resonance conditions uniformly in the wave number γ.

Trying to eliminate the quartic terms of the Hamiltonian obtained after the first canonical transformation, we see that there can be resonant terms: letting $a_\gamma = \sqrt{2}/2(\omega_\gamma^{-1/2}\eta_\gamma + i\omega_\gamma^{1/2}\Phi_\gamma)$, we see that all resonant terms are of the form $a_{\gamma_1}a_{\gamma_2}a^*_{\gamma_3}a^*_{\gamma_4}$, with the $\gamma_i = [l_i, m_i]$ satisfying the resonance conditions $\omega_{\gamma_1} + \omega_{\gamma_2} = \omega_{\gamma_3} + \omega_{\gamma_4}$ and $m_1 + m_2 = m_3 + m_4$,

$$\exists l \in \mathrm{Z}^+ \text{ s.t. } |l_1 - l_2| \le l \le l_1 + l_2 \text{ and } |l_3 - l_4| \le l \le l_3 + l_4.$$

We distinguish between two types of quartic resonant terms: first the (trivial) ones with $\gamma_3 = \gamma_1$ (or γ_2), $\gamma_4 = \gamma_2$ (or γ_1). There exist both integrable and nonintegrable terms of this type (the integrable ones have the form $|a_\gamma|^2|a_{\gamma'}|^2$). The second type of resonances are all other quartets satisfying the resonance conditions. A numerical search for resonances of this second type did not yield any, however, there is no bound away from such resonance that is uniform in γ. To define the quartic normal form Hamiltonian we may consider a certain cut-off k, and eliminate terms $a_{\gamma_1}a_{\gamma_2}a^*_{\gamma_3}a^*_{\gamma_4}$ for which $|\omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_3} - \omega_{\gamma_4}| \ge k$. Then denoting the remaining quartic terms by $\epsilon^2 N_{2,k}$, we can consider the quartic Hamiltonian $\bar{H}_k = H_0 + \epsilon^2 N_{2,k}$. Alternatively, we may work in R^{2n} with Galerkin approximations and consider the resulting quartic normal form Hamiltonian $\bar{H}_n = H_0 + \epsilon^2 N_{2,k}$. For up to $n = 350$ we can ascertain that $N_{2,n}$, contains only resonances of the first type. The absence of nontrivial resonances implies that axisymmetric motions corresponding to $\bar{H}_n$ are integrable. Using the above we can show that

Proposition 3.1. *For any mode index $\Gamma = [l_\Gamma, m_\Gamma]$ and $k > 0$, the plane V_Γ defined by $a_\gamma = 0$ for all $\gamma \neq \Gamma$ is invariant under the flow of the system (2.8) corresponding to $\bar{H}_k$.*

An analogous statement also holds for the Galerkin approximations: for any n, the system (2.8) corresponding to $\bar{H}_n$ has invariant planes V_Γ of $a_\gamma = 0$, for all $\gamma \neq \Gamma$. Both statements were shown in [2], under the assumption that the resonances of the second type are absent. However, we can see that this extra assumption is not necessary.

Near the origin, motion on the the invariant planes V_Γ is given by $a_\Gamma(t) = \mathrm{e}^{-i\Omega_\Gamma t}a_\Gamma(0)$, with $\Omega_\Gamma = \omega_\Gamma + \epsilon^2 C_\Gamma |a_\Gamma(0)|^2$, and $a_\gamma(t) \equiv 0$ for $\gamma \neq \Gamma$. The wave amplitude corresponding to these periodic orbits of Hamilton's equations with $\bar{H}_k$ (or $\bar{H}_n$) is

$$\eta(\vartheta, \varphi, t) = 2C_\Gamma |a(0)| P_{l_\Gamma}^{m_\Gamma}(\vartheta)\cos(m_\Gamma\varphi + \varphi_0 - \Omega_\gamma t), \tag{3.1}$$

with $P_{l_\Gamma}^{m_\Gamma}(\vartheta)$ the associated Legendre functions. Thus, these periodic orbits are traveling waves consisting of one spherical harmonic rotating about the north–south pole. Returning to the original variables, we see that we have near-monochromatic approximate solutions (more precisely solutions of approximating systems) given by (3.1) plus an O(ϵ) correction. Similarly, for $m_\Gamma = 0$ we have axisymmetric near-monochromatic standing waves.

In numerical experiments reported in [6] we saw that near-monochromatic motions are (locally) asymptotically stable, in the sense that trajectories starting near the periodic orbits of the systems corresponding to $\bar{H}_n$ stay nearby for long times. In the simulations we fixed $b = 1$ and $h = 0.2$, and considered Galerkin approximations of the cubic Hamiltonian $H = H_0 + \epsilon H_1$. The initial value problem for the Galerkin approximations was solved numerically using the numerical integration package LSODE. This simple spectral method is accurate (see [7] for details), but not very efficient and we only used it for cubic nonlinearity and limited spatial resolution. We obtained the results outlined below using the modes with $l \leq 8$, although, the theory we developed to explain them extends to larger Galerkin approximations and higher order nonlinearities.

We are interested in trajectories whose initial conditions contained one mode and small corrections, i.e. we set $a_\Gamma(0) = 1$, and $a_\gamma(0) < 10^{-2}$ for $\gamma \neq \Gamma$. We then measure the amplitude variation $\Delta A_\gamma(T)$, defined by

$$\Delta A_\gamma(T) \equiv \sup_{t\in[0,T]}|A_\gamma(t) - A_\gamma(0)|, \quad A_\gamma(t) = |a_\gamma(t)|, \tag{3.2}$$

over a time interval [0, T]. For a time interval [0, T] of about 250 periods (of the lowest modes with $l = 1$) we observe that

1. The mode Γ does not decay. Overall we have for $\epsilon = 10^{-4}$, $\Delta A_\Gamma(T) \in [10^{-6}, 10^{-5}]$; for $\epsilon = 10^{-3}$, $\Delta A_\Gamma(T)\tilde{1}0^{-4}$; for $\epsilon = 10^{-2}$, $\Delta A_\Gamma(T)\tilde{1}0^{-2}$.
2. Transfer of energy to other modes is limited. We generally observe that for $\epsilon = 10^{-4}$, $\Delta A_{\tilde{\gamma}}(T) \in [10^{-4}, 10^{-3}]$; for $\epsilon = 10^{-3}$, $\Delta A_{\tilde{\gamma}}(T) \in [10^{-3}, 10^{-2}]$; for $\epsilon = 10^{-2}$, $\Delta A_{\tilde{\gamma}}(T) \in [10^{-2}, 1.5 \times 10^{-2}]$. The transfer of energy to other modes can be appreciable as ϵ increases, but the mode Γ dominates. Also the $\Delta A_\gamma(T)$ is attained relatively early in our simulations (after about 50 periods) and are expected to be the same for longer observation times. Further numerical results as well as tests of their reliability can be found in [6].

4. Stability of near-monochromatic waves

The numerical simulations suggest that near-monochromatic motions posses some type of stability. The linear theory guarantees such stability for only a few periods and we thus, seek an explanation that takes nonlinearity into account. A partial explanation can be found by further studying the properties of the normal form equations and estimating the difference between such model systems and the equations solved numerically.

The main observation is that the finite dimensional normal form Hamiltonians $\bar{H}_n = H_0 + \epsilon N_{2,n}$ have additional constants of motion. In particular, assuming that the all quartic resonances are of the first type (this is verified in the truncations considered numerically), the Hamiltonian system corresponding to $\bar{H}_n = H_0 + \epsilon^2 N_{2,n} : R^{2n} \to R,\ n = l_{\max}^2 + 2l_{\max}$ has additional constants of motion I_l given by

$$I_l = \frac{1}{2l+1}\sum_{m=-l}^{l} |a_{[l,m]}|^2, \quad l = 1, 2, \ldots, l_{\max}. \tag{4.1}$$

The I_l are defined in terms of the variables obtained after the canonical transformations, so that returning to the original set of variables we can use the proposition and estimates for canonical transformations (e.g. [8]) to show ([6] for the precise statement) that

Proposition 4.1. *For $\epsilon > 0$ sufficiently small, and initial conditions in a neighborhood of size O(1) of the origin, the I_l, $l = 1, 2, \dots, l_{\max}$, satisfy*

$$|I_l(t) - I_l(0)| \le c\epsilon, \quad \forall t \in [0, \tilde{c}\epsilon^{-2}]. \tag{4.2}$$

The constants c, $\tilde{c}$ depend on the size of the Hamiltonian and are of O(1). Here the I_l are defined by (4.1), but in terms of the original coordinate system.

Therefore, if $|a_\Gamma(0)| = \mathrm{O}(1)$ and $|a_\gamma(0)| = \mathrm{O}(\epsilon)$ for $\gamma \neq \Gamma$, we expect that for $t \in [0, \tilde{c}\epsilon^{-2})$ we have $|a_\gamma(t)| = \mathrm{O}(\epsilon)$, $\forall \gamma = [l, m]$ with $l \neq l_\Gamma$. In the case of axisymmetric solutions the proposition gives a satisfactory explanation of the numerical observations. On the other hand, for general initial conditions we have the possibility of the amplitudes of modes with wave number $l = l_\Gamma$ changing to more than O(ϵ) in times greater than O(ϵ^{-1}). For shorter time scales stability to O(ϵ) follows from the absence of cubic resonances.

It is easy to see that in order to have to an O(ϵ^{-2}) stability time for general initial conditions, it is sufficient to have O(ϵ^{-2}) stability time for the Galerkin projections to the modes with the same wave number l. Such simplified systems also possess near-monochromatic periodic solutions, and we can study them in a more general setting. In particular, we consider analytic Hamiltonian systems of weakly coupled harmonic oscillators in R^{2n}. The coupling is O(ϵ) and we also assume that: (i) the linear frequencies are all equal, and that (ii) there exists a constant of motion $L_z = \sum_{j=-l}^{l} j|a_j|^2$, where the $j = -l, \dots, l$ index the oscillators (for an even number of oscillators we omit $j = 0$). The symmetry property (ii) comes from the water wave equations' symmetry under rotations around the north–south axis, and L_z is the analogue of angular momentum in the direction of that axis.

The resonance and symmetry properties of these Hamiltonian systems imply the existence of periodic orbits for which the amplitude of one mode can be O(1) and the amplitude of all other modes is O(ϵ) (see [9,10]). These orbits are analogous to the near-monochromatic orbits discussed above. In the systems under consideration the cubic part of the Hamiltonian can be eliminated by an analytic canonical transformation and this implies that trajectories starting in O(ϵ) neighborhoods of the periodic orbits stay in somewhat larger neighborhoods for an O(ϵ^{-1}) time. We want to examine the possibility of longer stability times, and in particular conditions for Nekhoroshev stability where the time of stability is $\sim\exp(\epsilon^{-\alpha})$, $\alpha > 0$, i.e. longer than any inverse power of ϵ as $\epsilon \to 0_+$. Existing results of this type (see [11,12]) assume that the Hamiltonian can be brought to a Birkhoff normal form with an integrable quartic part that is also convex in the actions. The integrability assumption is generic if the frequencies of the oscillators are left unspecified, but very special under the resonance property (i) above, and in fact does not hold for the projections of the water wave systems. Thus, we are particularly interested in understanding the effect of the lowest (here, quartic) nonintegrable terms of the Hamiltonian on the stability of the near-monochromatic periodic orbits.

To study this question we need some definitions. First, we consider a mode indexed by $j = \Gamma$ and the near-monochromatic periodic orbit $\nu_\Gamma \in R^{2n}$ with $|a_\Gamma(t)|$ of O(1) for all t identified above (viewed as an invariant set). Also, we eliminate cubic and nonresonant quartic terms in the Hamiltonian. In the quartic part of the resulting Birkhoff normal form Hamiltonian $\bar{H}$ we let $\epsilon\hat{h}_{\mathrm{BF}}(\Gamma)$ be the sum of monomials proportional to $a_{j_1}a_{j_2}a^*_\Gamma a^*_\Gamma$ (and complex conjugates) with $j_1, j_2 \neq \Gamma$. Also, we let $\epsilon\hat{h}_I(\Gamma)$ be the sum of monomials proportional to $|a_j|^2|a_\Gamma|^2$ with $j \neq \Gamma$. We have the following result.

Theorem 4.1. *Consider an analytic Hamiltonian system of coupled harmonic oscillators in R^{2n} satisfying conditions (i) and (ii), and the corresponding Birkhoff normal form Hamiltonian $\bar{H}$ as above (defined* *in a O(1) neighborhood of the origin for ϵ sufficiently small). Then every monochromatic periodic orbit $\nu_\Gamma \in R^{2n}$ has a tubular neighborhood of thickness O(ϵ^σ), $\sigma \in (0, 1/2)$ such that all trajectories starting in that neighborhood stay within a slightly thicker neighborhood for all $t \in (-T_m, T_m)$, where*

$$T_m = \tilde{c}_1 \epsilon^{-2} (M_2 + \tilde{M}_1 + \tilde{C}_{\mathrm{BF}} + \epsilon^\sigma \tilde{C}_I)^{-1}, \tag{4.3}$$

where

$$M_2 = K_1 \epsilon^{2+4\sigma} \exp(-K_2 \epsilon^{-\sigma/2}), \qquad \tilde{M}_1 = K_3 \epsilon^4 \exp(-K_4 \epsilon^{-1}). \tag{4.4}$$

$\tilde{C}_{\mathrm{BF}}$, $\tilde{C}_I$ are the supremum norms of $\hat{h}_{\mathrm{BF}}(\Gamma)$, $\hat{h}_I(\Gamma)$, respectively in a O(1) neighborhood of the origin. The constants $\tilde{c}_1$, K_1, K_2, K_3, K_4 are of O(1) except in the degenerate (or trivial) case where the product of the coefficient C_Γ of $\epsilon |a_\Gamma|^4$ and $|a_\Gamma(0)|$ is small.

The precise statement can be found in [9], here we only want to capture the essential points. In particular, if $\hat{h}_{\mathrm{BF}}(\Gamma)$ and $\hat{h}_I(\Gamma)$ vanish we have Nekhoroshev stability of the orbits. In the case where $\hat{h}_{\mathrm{BF}}(\Gamma)$ is small, e.g. $\tilde{C}_{\mathrm{BF}}$ is O(ϵ^σ), we have an O($\epsilon^{-(2+\sigma)}$) stability time for an O(ϵ^σ) neighborhood of the periodic orbit. In the generic case where $\tilde{C}_{\mathrm{BF}}$ is O(1) we have an O(ε^{-2}) stability time for an O(ε^σ) neighborhood of ν. In both cases the stability time is improved although, we consider larger neighborhoods. However, it seems that Nekhoroshev stability is obtained only under very strong assumptions.

The proof of Theorem 4.1 follows a strategy of [13] for showing Nekhoroshev stability in perturbed NLS equations, and uses Birkhoff normal forms around the origin and periodic orbits to identify slowly evolving local action-like variables that control the distance from the periodic orbits. Our work adds careful estimates of the quartic nonintegrable terms, and we note that the assumptions can be checked quite easily since they only involve the quartic terms of $\bar{H}$.

5. Discussion

From the above we see that we can obtain more refined information on asymptotic stability of the near monochromatic waves seen numerically. In the last proposition we saw that certain integrable terms denoted by $\hat{h}_I(\Gamma)$ may be an obstacle to Nekhoroshev stability. As we discuss in [9], if the integrable quartic terms are convex in the actions the constant $\tilde{C}_I$ will not appear in (4.3), and we believe that the condition on the absence of $\hat{h}_I(\Gamma)$ can be replaced by a condition of convexity of the integrable quartic terms in a neighborhood of the periodic orbit. Such a condition is not weaker, but it has a more geometrical meaning.

On the other hand the main goal has been to understand the effect of nonintegrable terms on the stability of the periodic orbits and we identified the quartic terms denoted by $\hat{h}_{BF}(\Gamma)$ as the main obstacle. Similar terms may cause linear instability of Stokes waves in deep water waves (see [14]) and other dispersive wave equations. The linear instability is due to sub-harmonic parametric excitation of the modes with very small amplitude, and the (asymptotic) theory of the Mathieu equation suggests that it can be avoided provided that the ratio of the size of the $\hat{h}_{\mathrm{BF}}(\Gamma)$ terms coupling the dominant mode Γ to other modes to the product $C_\gamma |a_\Gamma(0)|^2$ satisfies a certain bound (this is a “coupling-to-detuning” ratio). Following an approach that is similar to the one used to obtain Theorem 4.1 we can see that this condition improves the stability time, but only by O($\epsilon^{-4\sigma}$).

These comments suggest that the asymptotic stability results we presented can be improved, and work on these improvements is in progress. In any case, we have seen that the use of Hamiltonian perturbation methods can give useful information on the dynamics of water waves, although, so far the rigorous results we presented here concern Galerkin truncations. Thus, the possibility of extending some of these results to infinite dimensional water wave models is a question of significant interest.

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