LEMMA 2. *Under the same conditions in Lemma* 1 *we have that* Φ' *is R-integrable over* $[a, b]$.

Proof. Lemma 1 is applicable on every closed subinterval of $[a, b]$. From this it follows that the fluctuation of Φ' is not larger than the fluctuation of ϕ.

According to a well-known criterion for R-integrability [**2**, p. 107] we may conclude that Φ' is R-integrable over $[a, b]$.

Proof of the theorem. It is clear that the given condition is necessary; take $\phi = f'$.

To prove the sufficiency we write

$$f(x) - f(a) = \int_a^x \phi(t)dt.$$

Then it is clear that f' is the derivative of a function of the form $\int_a^x \phi(t)dt$. According to Lemma 2 we obtain that f' is R-integrable over $[a, b]$, completing the proof.

REMARK. Clearly we have

$$\int_a^x f'(t)dt = \int_a^x \phi(t)dt, \qquad \forall\, x \in [a, b].$$

However, it is easy to construct examples in which we do not have $f' = \phi$.

In general ϕ may have simple discontinuities, whereas, according to a theorem of Darboux, a derivative can only have discontinuities of the second kind. Compare [**2**, p. 94].

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A SIMPLE PROOF FOR PARTIAL PIVOTING

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Let A be a (real), $n \times n$, nonsingular matrix and suppose we wish to solve numerically the linear system of equations

$$Ax = b. \tag{1}$$

This task is straightforward if we can factor A as $A = LU$ where $L = (l_{ij})$ is lower triangular ($l_{ij} = 0$ for $i < j$) and $U = (u_{ij})$ is upper triangular ($u_{ij} = 0$ for $i > j$) since then we solve

$$Ly = b \tag{2}$$

and

$$Ux = y, \tag{3}$$

obtaining first $y_1, y_2, y_3, \cdots, y_n$ from (2) and finally $x_n, x_{n-1}, x_{n-2}, \cdots, x_1$ from (3). Unfortunately, it is not always possible to factor $A = LU$ as above; take, for example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The purpose of this note is to offer a simple inductive (and constructive) proof of the following fundamental theorem of numerical linear algebra.

THEOREM. *For any nonsingular $n \times n$ matrix, A, there exists an $n \times n$ permutation matrix P such that*

$$PA = LU,$$

where L is lower triangular with ones on the diagonal and U is upper triangular with nonzero diagonal elements.

Proofs of the "$PA = LU$" theorem are offered by relatively few authors of numerical analysis texts. Wendroff [**3**, pp. 127–129] presents a complete proof which is similar in spirit to Wilkinson's sketch in [**4**, pp. 206–207]. Stewart [**2**, Chapt. 3, §2] gives a proof as part of his general discussion of Gaussian elimination while Forsythe and Moler [**1**, Chapt. 16, pp. 63–64] imbed a proof in their algorithm and subsequent discussion.

Suppose A is written as

$$A = \begin{bmatrix} a & r \\ c & B \end{bmatrix} \tag{4}$$

where $a \neq 0$ is scalar, c is $(n-1)\times 1$, r is $1 \times (n-1)$ and B is $(n-1)\times(n-1)$. Most authors view Gaussian elimination as effecting elementary row operations on A and its transforms until one obtains an upper triangular matrix. For example, the first step takes $A \equiv A^{(1)}$ into $A^{(2)}$ by

$$L_1 A^{(1)} = \begin{bmatrix} a & r \\ 0 & B^{(2)} \end{bmatrix} = A^{(2)},$$

where

$$L_1 = \begin{bmatrix} 1 & 0 \\ -c/a & I \end{bmatrix}, \qquad B^{(2)} = B - cr/a.$$

One then continues (formally) finding elementary lower triangular matrices ([**2**], p. 115), L_i such that

$$L_i L_{i-1} \cdots L_1 A = A^{(i+1)} \equiv (a_{ij}^{(i+1)}) \tag{5}$$

has zero below the diagonal in its first i columns. $A^{(n)}$ is then upper triangular, $L = L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1}$ is lower triangular so $A = LU$. The hitch in this formal discussion is that possibly some $a_{ii}^{(i)} = 0$ so, in general, (5) must be replaced by an expression with permutation matrices sandwiched in between the L_i (see [**3**, p. 128] or [**2**, p. 124]); this causes some untidyness. An alternative formal derivation of $A = LU$ proceeds from (4) to

$$A = \begin{bmatrix} 1 & 0 \\ c/a & I \end{bmatrix} \begin{bmatrix} a & r \\ 0 & B^{(2)} \end{bmatrix} \tag{6}$$

and supposing $B^{(2)} = L_2 U_2$ to

$$A = \begin{bmatrix} 1 & 0 \\ c/a & L_2 \end{bmatrix} \begin{bmatrix} a & r \\ 0 & U_2 \end{bmatrix} = LU.$$

There is nothing "existential" about this argument; one would begin to factor $B^{(2)}$ exactly as one began to factor A itself.

Proof of Theorem. The proof of the theorem is shorter than its motivation so let us begin. The nonsingular matrix A must have a nonzero in its first column; hence there exists a permutation matrix P_1 such that

$$P_1A=\begin{bmatrix} a & r \\ c & B \end{bmatrix}, \qquad a\neq 0,$$

as in (4). Then as in (6)

$$P_1A=\begin{bmatrix} 1 & 0 \\ c/a & I \end{bmatrix}\begin{bmatrix} a & r \\ 0 & B^{(2)} \end{bmatrix},$$

$B^{(2)}=B-cr/a$. Now $B^{(2)}$ is nonsingular so by induction (the 1×1 case is clear) there exists an $(n-1)\times(n-1)$ permutation matrix $\bar{P}_2$ such that $\bar{P}_2B^{(2)}=L_2U_2$. Thus

$$P_1A=\begin{bmatrix} 1 & 0 \\ c/a & \bar{P}_2^T \end{bmatrix}\begin{bmatrix} a & r \\ 0 & L_2U_2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ c/a & \bar{P}_2^TL_2 \end{bmatrix}\begin{bmatrix} a & r \\ 0 & U_2 \end{bmatrix}$$

and

$$P_2P_1A=\begin{bmatrix} 1 & 0 \\ \bar{P}_2c/a & L_2 \end{bmatrix}\begin{bmatrix} a & r \\ 0 & U_2 \end{bmatrix}, \quad \text{where} \quad P_2=\begin{bmatrix} 1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix}. \tag{7}$$

Hence $PA=LU$ as in the theorem.

If algorithms can contain proofs, then perhaps proofs can also contain algorithms. Notice that in our algorithm the $\bar{P}_2c/a$ expression in (7) is important. Its presence says that as we overwrite A by successive columns of L and rows of U, any row interchanges on the subsequent submatrices must in fact be done on the whole evolving matrix. (One can also avoid any physical interchanging as discussed in [**1**, Chapt. 16].) Finally, when executing our proof on any modern day computer with inexact arithmetic (and therefore round-off error), choose a (and subsequent "pivots") to be the element of largest magnitude in the column (on or below the diagonal). The proof is then called "Gaussian elimination with partial pivoting." We refer to [**1**] or [**2**] for more extensive discussions and to [**4**] for the classic on the subject.

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