Modulational and spectral (in)stability of periodic traveling wave solutions to the nonlinear Klein-Gordon equation

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The nonlinear Klein-Gordon equation

Nonlinear Klein-Gordon with periodic potential:

$$u_{tt} - u_{xx} + V'(u) = 0. \qquad \text{(nKG)}$$

for $(x,t) \in \mathbb{R} \times [0, +\infty)$, u scalar, $V \in C^2$, periodic.

Sine-Gordon equation:

$$u_{tt} - u_{xx} + \sin u = 0. \qquad \text{(SG)}$$

$V(u) = 1 - \cos u.$

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Applications (sine-Gordon):

- Surfaces with negative Gaussian curvature (Eisenhart, 1909)
- Propagation of crystal dislocations (Frenkel and Kontorova, 1939)
- Elementary particles (Perring and Skyrme, 1962)
- Propagation of magnetic flux on a Josephson line (Scott, 1969)
- Dynamics of fermions in the Thirring model (Coleman, 1975)
- Oscillations of a rigid pendulum attached to a stretched rubber band (Drazin, 1983)

Assumptions on the potential:

(a) $V : \mathbb{R} \to \mathbb{R}$ is of class C^2 in all its domain and it is periodic with fundamental period P.

(b) V has only non-degenerate critical points.

(c) $V'(u)^4 (V(u)/V'(u)^2)'' \geq 0$ for all u under consideration.

Assumption (c) implies monotonicity of the period map with respect to the energy.

Traveling waves

$u(x,t) = f(x-ct)$, $z = x - ct$, solution to the nonlinear pendulum equation:

$$(c^2-1)f_{zz} + V'(f(z)) = 0,$$

Sine-Gordon case:

$$(c^2-1)f_{zz} + \sin(f(z)) = 0,$$

$c \in \mathbb{R}$ (wave speed), $c^2 \neq 1$.

Upon integration:

$$\frac{1}{2}(c^2-1)f_z^2 = E - V(f),$$

E = constant (energy). Under assumptions:
$0 < E_0 = \max V(u)$

Sine-Gordon case: $V(u) = 1 - \cos u$, $E_0 = 2$,

$$\frac{1}{2}(c^2-1)f_z^2 = E - 1 + \cos f(z).$$

Classification

First dichotomy (wave speed):

- **Subluminal** waves: $c^2 < 1$
- **Superluminal** waves: $c^2 > 1$

Second dichotomy (energy E):

- **Librational** wavetrain: $f(z+T) = f(z)$. Closed trajectory inside the separatrix in the phase portrait.
- **Rotational** wavetrain: $f(z+T) = f(z) \pm P$. Open trajectory outside the separatrix in the phase plane. Sign f_z is fixed.

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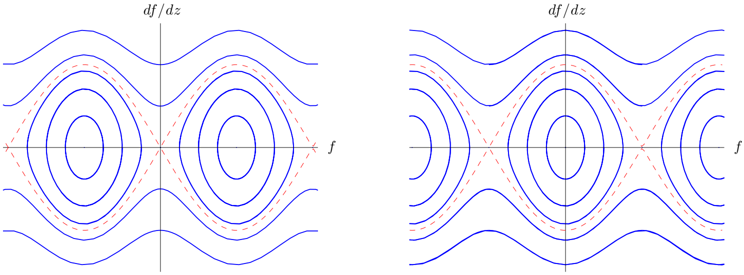

Figure: Phase portrait: subluminal (left); superluminal (right).

Superluminal librational: $c^2 > 1$, $0 < E < E_0$.
$\mathcal{K}(E) = \{u \in \mathbb{R} : (E - V(u))/(c^2 - 1) \geq 0\}$ = disjoint union of intervals in $(0, P)$. In (v_1, v_2), only one non-degenerate zero of V'. Librational (closed) periodic orbit.

$$f_z = \frac{\sqrt{2}}{\sqrt{c^2-1}} \sqrt{E - V(f)},$$

where $f \in (v_1, v_2) \subset \mathcal{K}(E)$.

$$T = \sqrt{2}\sqrt{c^2-1} \int_{v_1}^{v_2} \frac{d\eta}{\sqrt{E - V(\eta)}}.$$

Sine-Gordon: wave oscillates around $f = 0$, in $(v_1, v_2) = (-\operatorname{Arc}\cos(-E+1), \operatorname{Arc}\cos(-E+1))$

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$\mathcal{K}(E) = \{u \in \mathbb{R} : (V(u) - E)/(1 - c^2) \geq 0\}$ = disjoint union of intervals in $(0, P)$. In (v_3, v_4), only one non-degenerate zero of V'. Librational (closed) periodic orbit.

$$f_z = \frac{\sqrt{2}}{\sqrt{1-c^2}}\sqrt{V(f) - E},$$

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Superluminal rotational: $c^2 > 1$, $E > E_0$, $E - V(f) > 0$ and $\mathcal{K}(E) = \mathbb{R}$. Rotation, f_z has fixed sign. Orbit outside the separatrix and $f(z+T) = f(z) \pm P$ for all z.

$$f_z^2 = \frac{2(E - V(f))}{c^2 - 1} > 0,$$

$$T = \frac{\sqrt{c^2-1}}{\sqrt{2}} \int_0^P \frac{d\eta}{\sqrt{E - V(\eta)}}$$

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$$\mathcal{R}_1 = \{c^2 < 1,\ 0 < E < E_0\},$$ (subluminal librational),

$$\mathcal{R}_2 = \{c^2 < 1,\ E < 0\},$$ (subluminal rotational),

$$\mathcal{R}_3 = \{c^2 > 1,\ 0 < E < E_0\},$$ (superluminal librational),

$$\mathcal{R}_4 = \{c^2 > 1,\ E > E_0\},$$ (superluminal rotational),

$$(E,c) \in \mathcal{R} = \cup_{j=1}^{4} \mathcal{R}_j$$

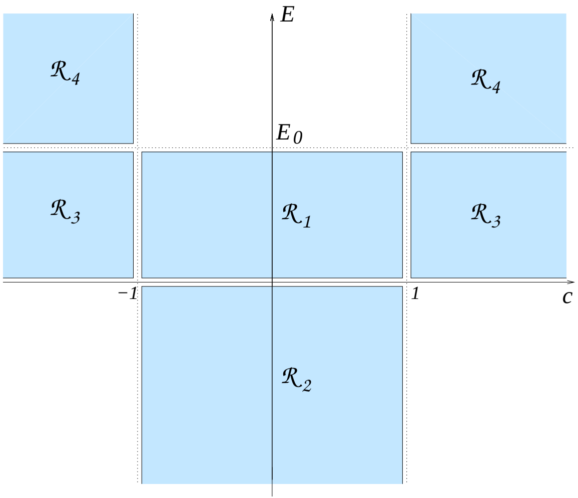

Figure: Sketch of the open set $\mathcal{R} \subset \mathbb{R}^2$.

Spectral problem

Solution $f(z)+u(z,t)$, with $u=$ perturbation:

$$u_{tt}-2cu_{zt}+(c^2-1)u_{zz}+V'(u+f)-V'(f)=0.$$

Linearized equation:

$$u_{tt}-2cu_{zt}+(c^2-1)u_{zz}+V''(f(z))u=0.$$

$u=w(z)e^{\lambda t}$, $\lambda\in\mathbb{C}$, $w\in X$ Banach:

$$(c^2-1)w_{zz}-2c\lambda w_z+(\lambda^2+V''(f(z)))w=0. \qquad \text{(P)}$$

Quadratic “pencil” in λ.

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Quadratic “pencil” in λ.

Floquet spectrum

$\lambda \in \sigma_F$ is a Floquet eigenvalue if there exists a bounded solution w to (P).

We say the wave is *spectrally stable* if $\sigma_F \subset \{\mathrm{Re}\,\lambda > 0\}$. Otherwise it is *spectrally unstable.*

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Previous results

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Summary of stability results

Wave	Whitham (1974)	Scott (1969)
Subluminal rotational	stable	stable
Superluminal rotational	stable	unstable
Subluminal librational	unstable	unstable
Superluminal librational	unstable	unstable

Whitham (1965, 1974):

Modulation theory: well established (formal) physical method based on WKB expansions. Exact wave $f = f(x - ct) = \tilde{f}(kx - \omega t)$. Allowing dependence $k = k(x,t), \omega = \omega(x,t)$, under "slow modulations", if the PDE system on (k,ω) is well-posed then the wave is "stable".

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Scott (1969):

$$y = \exp\left(\frac{-c\lambda z}{c^2-1}\right),$$

$$y_{zz} + \frac{V''(f(z))}{c^2-1} y = \left(\frac{\lambda}{c^2-1}\right)^2 y =: \nu y. \qquad \text{(H)}$$

Hill's equation with period T. $\nu \in \sigma_H$ (Floquet spectrum of (H)) if there is a bounded solution y.

Scott assumed that the transformation is *isospectral*. This is not true. Actually:

Lemma (JMMP1)

If $\lambda \in \sigma_H \cap \sigma_F$ then $\lambda \in i\mathbb{R}$.

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References:

- C.K.R.T. Jones, R. Marangell, P.D. Miller, R.P., *On the stability of periodic traveling sine-Gordon waves*. Preprint, 2012. arXiv:1210.0659. **(JMMP1)**.
- C.K.R.T. Jones, R. Marangell, P.D. Miller, R.P., *Modulational and spectral (in)stability of periodic wavetrains for the nonlinear Klein-Gordon equation.* Preprint, 2012. **(JMMP2)**.

Spectrum revisited. Evans function.

Problem (P) can be written as a first order system:

$$W_z = \mathbb{A}(z,\lambda)W,$$

$$W := \begin{pmatrix} w \\ w_z \end{pmatrix},$$

$$\mathbb{A}(z,\lambda) := \begin{pmatrix} 0 & 1 \\ -\dfrac{(\lambda^2 + V''(f(z)))}{c^2-1} & \dfrac{2c\lambda}{c^2-1} \end{pmatrix}.$$

Family of closed, densely defined operators:

$$\mathcal{T}(\lambda) : \mathcal{D} \subset X \to X$$

$$\mathcal{T}(\lambda)W := W_z - \mathbb{A}(z,\lambda)W.$$

E.g.:

$$\mathcal{D} = H^1(\mathbb{R};\mathbb{C}^2), \quad X = L^2(\mathbb{R};\mathbb{C}^2),$$

Spectral stability of periodic waves with respect to *localized perturbations*.

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Definition (Sandstede (2002))

The *resolvent* ζ, the *point spectrum* σ_{pt} and the *essential spectrum* σ_{ess} of problem (P) are defined as

$$\begin{aligned}\zeta &:= \{\lambda \in \mathbb{C} : \mathcal{T}(\lambda) \text{ is one-to-one and onto, and} \\ &\quad \mathcal{T}(\lambda)^{-1} \text{ is bounded}\},\\ \sigma_{\text{pt}} &:= \{\lambda \in \mathbb{C} : \mathcal{T}(\lambda) \text{ is Fredholm with zero index} \\ &\quad \text{and has a non-trivial kernel}\},\\ \sigma_{\text{ess}} &:= \{\lambda \in \mathbb{C} : \mathcal{T}(\lambda) \text{ is either not Fredholm or} \\ &\quad \text{has index different from zero}\}.\end{aligned}$$

The *spectrum* is $\sigma = \sigma_{\text{ess}} \cup \sigma_{\text{pt}}$. ($\mathcal{T}(\lambda)$ closed $\Rightarrow \zeta = \mathbb{C} \backslash \sigma$.)

Lemma

All spectrum of problem (P) *is "continuous", that is,* $\sigma = \sigma_{ess}$ *and* σ_{pt} *is empty.*

Monodromy matrix:

$$\mathbb{M}(\lambda) := \Phi(T, \lambda)$$

$\Phi(z, \lambda) =$ fundamental solution with $\Phi(0, \lambda) = \mathbb{I}$.

$$\mathbb{M}(\lambda)\Phi(z, \lambda) = \Phi(z + T, \lambda)$$

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Floquet multipliers:

$\lambda \in \sigma$ if and only if there exists at least one $\mu \in \mathbb{C}$ (Floquet multiplier) with $|\mu| = 1$ such that

$$\hat{D}(\lambda, \mu) := \det(\mathbb{M}(\lambda) - \mu \mathbb{I}) = 0.$$

$\mu = \mu(\lambda) = e^{i\theta(\lambda)}$ are the eigenvalues of $\mathbb{M}(\lambda)$. $\theta = \theta(\lambda)$ are called the Floquet exponents.

Periodic Evans function (Gardner, 1997):

Definition

The *periodic Evans function* $D : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is

$$D(\lambda, \kappa) := \hat{D}(\lambda, e^{i\kappa T}) = \det(\mathbb{M}(\lambda) - e^{i\kappa T}\mathbb{I}),$$

for each $(\lambda, \kappa) \in \mathbb{C} \times \mathbb{R}$.

Properties: (Gardner 1997, 1998)

- σ is the set of all $\lambda \in \mathbb{C}$ such that $D(\lambda, \kappa) = 0$ for some real κ.
- D is analytic in λ and κ.
- The order of the zero in λ is the multiplicity of the eigenvalue.
- $\hat{D}(\lambda, 1) = D(\lambda, 0)$ detects spectra corresponding to perturbations which are T-periodic.

Floquet spectrum:
Boundary value problem of the form

$$(c^2-1)w_{zz}-2c\lambda w_z+(\lambda^2+V''(f(z)))w=0,$$

$$\begin{pmatrix} w(T) \\ w_z(T) \end{pmatrix} = e^{i\theta} \begin{pmatrix} w(0) \\ w_z(0) \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

For a given $\theta \in \mathbb{R}$ we define $\sigma_\theta \subset \mathbb{C}$ to be the set of complex λ for which there exists a nontrivial solution. The Floquet spectrum σ_F is defined then as the union over θ of these sets:

$$\sigma_F := \bigcup_{-P<\theta\leq P} \sigma_\theta.$$

Clearly: $\sigma = \sigma_F$.

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Solutions at $\lambda = 0$

$f = f(z; E, c)$, $(E, c) \in \mathcal{R}$. Initial conditions:

$$u_0(E,c) = f(0;E,c)$$
$$= \begin{cases} f(T;E,c), & E \in (0, E_0), \quad \text{(lib)}, \\ f(T;E,c) - P, & E \in (-\infty, 0) \cup (E_0, +\infty), \quad \text{(rot)}, \end{cases}$$

$$v_0(E,c) = f_z(0;E,c) = f_z(T;E,c)$$

System at $\lambda = 0$:

$$Y_z = \mathbb{A}(z,0)Y,$$

$$\mathbb{A}(z,0) = \begin{pmatrix} 0 & 1 \\ -V''(f(z))/(c^2-1) & 0 \end{pmatrix}.$$

Lemma

The two-dimensional vector space of solutions is spanned by

$$Y_0(z) = \begin{pmatrix} f_z \\ f_{zz} \end{pmatrix}, \quad \textit{and} \quad Y_1(z) = \begin{pmatrix} f_E \\ f_{Ez} \end{pmatrix}.$$

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$$\det(Y_0(z), Y_1(z)) = f_z f_{Ez} - f_E f_{zz} = \frac{1}{c^2 - 1} \neq 0$$

Solution matrix:

$$\mathbb{Q}(z,0) := (Y_0(z), Y_1(z))$$

$$\Phi(z,0) = \mathbb{Q}(z,0)\mathbb{Q}(0,0)^{-1}.$$

$$\mathbb{M}(0) = \Phi(T,0) = \mathbb{Q}(T,0)\mathbb{Q}(0,0)^{-1}$$

$$\mathbb{Q}(z,0)^{-1} = (c^2 - 1)\begin{pmatrix} f_{Ez} & -f_E \\ -f_{zz} & f_z \end{pmatrix}.$$

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Lemma

If $T_E \neq 0$, there exists a basis in $\mathbb{R}^2$ such that the monodromy map $\mathbb{M}(\lambda)$ at $\lambda = 0$ has the Jordan form

$$\mathbb{M}(0) \sim \begin{pmatrix} 1 & -T_E \\ 0 & 1 \end{pmatrix}.$$

$\mathbb{Q}(T,0) - \mathbb{Q}(0,0)$ is a rank-one matrix provided that $T_E \neq 0$:

$$\mathbb{Q}(T,0) = \mathbb{Q}(0,0) + \begin{pmatrix} 0 & -T_E v_0(E,c) \\ 0 & -T_E \frac{V'(u_0(E,c))}{c^2-1} \end{pmatrix}$$

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Under Assumption (c), we have monotonicity of the period map (Chicone, 1987: criterion for planar Hamiltonian systems):

Lemma

Under assumptions there holds $T_E \neq 0$. More precisely we have:

(i) $T_E > 0$ *in the rotational subluminal and librational superluminal cases.*

(ii) $T_E < 0$ *in the rotational superluminal and librational subluminal cases.*

Lemma

If we define

$$\bar{\Delta} := -\frac{T_E}{c^2 - 1}$$

then

(a) $\bar{\Delta} > 0$ *for rotational waves.*

(b) $\bar{\Delta} < 0$ *for librational waves.*

Solutions series expansions

$\mathbb{Q} = \mathbb{Q}(z,\lambda)$ solution to

$$\frac{d\mathbb{Q}}{dz} = \mathbb{A}(z,\lambda)\mathbb{Q}.$$

$$\mathbb{Q}(0,\lambda) = \mathbb{Q}(0,0) = (Y_0(0),\, Y_1(0))$$

By analyticity, seek series expansion

$$\mathbb{Q}(z,\lambda) = \sum_{n=0}^{+\infty} \lambda^n \mathbb{Q}_n(z)$$

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Collecting like powers of λ we obtain a hierarchy:

$$(c^2-1)\frac{d\mathbb{Q}_1}{dz} = \mathbb{A}_0(z)\mathbb{Q}_1 + \mathbb{A}_1\mathbb{Q}_0$$

$$(c^2-1)\frac{d\mathbb{Q}_n}{dz} = \mathbb{A}_0(z)\mathbb{Q}_n + \mathbb{A}_1\mathbb{Q}_{n-1} + \mathbb{A}_2\mathbb{Q}_{n-2}, \;\; n = 2,3,\ldots$$

Solution by variation of parameters:

$$\mathbb{Q}_1(z) = \frac{\mathbb{Q}_0(z)}{c^2-1}\int_0^z \mathbb{Q}_0(y)^{-1}\mathbb{A}_1\mathbb{Q}_0(y)\,dy$$

$$\mathbb{Q}_n(z) = \frac{\mathbb{Q}_0(z)}{c^2-1}\int_0^z \mathbb{Q}_0(y)^{-1}\big(\mathbb{A}_1\mathbb{Q}_{n-1}(y) + \mathbb{A}_2\mathbb{Q}_{n-2}\big)\,dy, \;\; n \geq 2$$

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By Abel's identity:

Lemma

For all $z \in \mathbb{R}$, $\lambda \in \mathbb{C}$, *there holds*

$$\det \mathbb{Q}(z,\lambda) = \frac{\exp\left(2c\lambda z/(c^2-1)\right)}{c^2-1}.$$

After (tedious) computations:

Lemma

$$\operatorname{tr} \mathbb{Q}_0(T)\mathbb{Q}_0(0)^{-1} = 2.$$

$$\operatorname{tr} \mathbb{Q}_1(T)\mathbb{Q}_0(0)^{-1} = \frac{2cT}{c^2 - 1}.$$

$$\operatorname{tr} \mathbb{Q}_2(T)\mathbb{Q}_0(0)^{-1} = \frac{c^2T^2}{(c^2-1)^2} - \frac{T_E}{c^2 - 1}\int_0^T f_z(y)^2\,dy.$$

Perturbation of the Jordan block

By analyticity of the monodromy map:

$$\mathbb{M}(\lambda) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} \frac{d^n \mathbb{M}}{d\lambda^n}(0).$$

(Standard perturbation theory, Kato.) In general, the Floquet multipliers bifurcate from $\lambda = 0$ in Pusieux series.

Fundamental matrix:

$$\Phi(z,\lambda) = \mathbb{Q}(z,\lambda)\mathbb{Q}_0(0)^{-1} = \sum_{n=0}^{+\infty} \lambda^n \mathbb{Q}_n(z)\mathbb{Q}_0^{-1} =: \sum_{n=0}^{+\infty} \lambda^n \Phi_n(z)$$

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Lemma

We have convergent series expansions

$$\mathbb{M}(\lambda) = \sum_{n=0}^{+\infty} \lambda^n \mathbb{Q}_n(T)\mathbb{Q}_0(0)^{-1},$$

$$\operatorname{tr}\mathbb{M}(\lambda) = \sum_{n=0}^{+\infty} \lambda^n \operatorname{tr}\mathbb{Q}_n(T)\mathbb{Q}_0(0)^{-1},$$

and $$\det\mathbb{M}(\lambda) = \sum_{n=0}^{+\infty} \left(\frac{2cT}{c^2-1}\right)^n \frac{\lambda^n}{n!},$$

Expansion of the Floquet multipliers

μ, solutions to:

$$\hat{D}(\lambda,\mu) = \det(\mathbb{M}(\lambda) - \mu\mathbb{I}) = \mu^2 - (\operatorname{tr}\mathbb{M}(\lambda))\mu + \det\mathbb{M}(\lambda) = 0$$

$$\mu_{\pm}(\lambda) = \frac{1}{2}\Big(\operatorname{tr}\mathbb{M}(\lambda) \pm \big((\operatorname{tr}\mathbb{M}(\lambda))^2 - 4\det\mathbb{M}(\lambda)\big)^{1/2}\Big)$$

Expanding:

$$\operatorname{tr}\mathbb{M}(\lambda)^2 - 4\det\mathbb{M}(\lambda) =$$

$$\left(\operatorname{tr}\mathbb{Q}_0(T)\mathbb{Q}_0(0)^{-1} + \lambda\operatorname{tr}\mathbb{Q}_1(T)\mathbb{Q}_0(0)^{-1} + \lambda^2\operatorname{tr}\mathbb{Q}_2(T)\mathbb{Q}_0(0)^{-1}\right)^2 +$$

$$-4\left(1 + \frac{2cT}{c^2-1}\lambda + \frac{2c^2T^2}{(c^2-1)^2}\lambda^2\right) + O(\lambda^3)$$

$$= 4\Delta\lambda^2 + O(\lambda^3),$$

$$\Delta := -\frac{T_E}{c^2-1}\int_0^T f_z(y)^2\,dy$$

The two Floquet multipliers are analytic functions of λ at $\lambda = 0$. Asymptotic form:

$$\mu_{\pm}(\lambda) = 1 + \Big(\frac{cT}{c^2-1} \pm \Delta^{1/2}\Big)\lambda + O(\lambda^2)$$

Definition

We define the *modulational instability index* to be the quantity

$$\rho := \operatorname{sgn}\Delta.$$

Clearly $\operatorname{sgn}\Delta = \operatorname{sgn}\bar{\Delta}$.

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Expansion of D near the origin

Lemma

The periodic Evans function $D(\lambda,\kappa)$, for $(\lambda,\kappa)\in\mathbb{C}\times\mathbb{R}$, has the following expansion in a neighborhood of $(\lambda,\kappa)=(0,0)$,

$$D(\lambda,\kappa) = -\Delta\lambda^2 + \Big(i\kappa - \frac{cT}{c^2-1}\lambda\Big)^2 + O(3),$$

where $O(3)$ denotes terms of order three or higher in (λ,k).

Lemma

If $\rho = 1$ *then the solutions to* $D(\lambda, \kappa) = 0$ *near* $(\lambda, \kappa) = (0, 0)$ *emerge from the origin tangentially to the imaginary axis in the complex* λ*-plane:*

$$\lambda(\kappa) = -iv\kappa + O(\kappa^2),$$

with $v \in \mathbb{R}$*, for* $|\kappa| \ll 1$*.*
If $\rho = -1$ *then the solutions emerge from the origin tangentially to two lines passing through the origin and forming non-zero angles with the imaginary axis:*

$$\lambda(\kappa) = -(\alpha + i\beta)\kappa + O(\kappa^2),$$

with $\alpha, \beta \in \mathbb{R}$*,* $\alpha \neq 0$*, for* $|\kappa| \ll 1$*.*

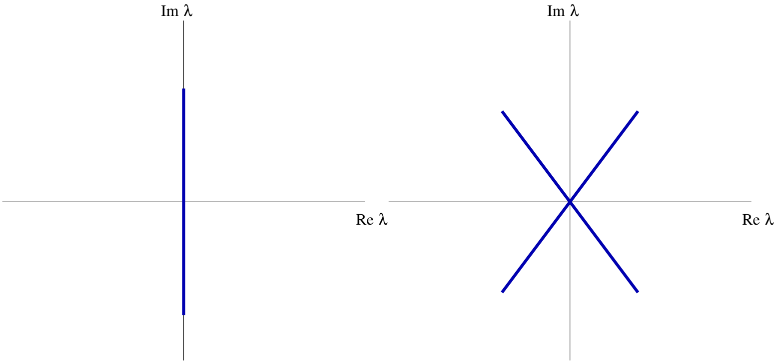

Figure: Qualitative sketch of σ near the origin. $\rho = 1$ (left); $\rho = -1$ (right).

Theorem

Under assumptions (a), (b) and (c):

- $\rho = -1$ *for librational waves. Spectrally unstable.*
- $\rho = 1$ *for rotational waves. The spectrum is tangent to the imaginary axis at* $\lambda = 0$.

Theorem

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Relation to Whitham's modulation theory

Reference: Whitham, Proc. Roy. Soc. Ser. A (1965).

WKB approximations of the form:

$$u(x,t) = f\left(\frac{z(x,t)}{\varepsilon}\right) + O(\varepsilon),$$

k, ω are no longer constant (and hence, E and c). We have $c = \omega/k$ and $k = \theta_x$, $\omega = -\theta_t$, $\theta = kx - \omega t$. Conservation of fluxons:

$$k_t + \omega_x = 0$$

Averaged Lagrangian

$$I[u] = \int\int L(u, u_x, u_t)\, dx\, dt,$$

$$L(u, u_x, u_t) = \tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_x^2 - V(u).$$

In the wave $u = f(x - ct) = \Phi(kx - \omega t)$:

$$L(u, u_x, u_t) = \tfrac{1}{2}(\omega^2 - k^2)\Phi_\theta(\theta)^2 - V(\Phi(\theta))$$

Averaged Lagrangian:

$$\langle L \rangle = \frac{1}{kT}\int_0^{kT} \tfrac{1}{2}(\omega^2 - k^2)\Phi_\theta(\theta)^2 - V(\Phi(\theta))\, d\theta = \tilde{\mathcal{L}}(\omega, k, E).$$

Averaged Lagrangian variational principle

$$\delta \iint \tilde{\mathcal{L}}(\omega, k, E)\, dx\, dt = 0,$$

$\tilde{\mathcal{L}}_E = 0,$ dispersion relation

$$\begin{aligned} k_t + \omega_x &= 0 \\ (\tilde{\mathcal{L}}_\omega)_t - (\tilde{\mathcal{L}}_k)_x &= 0. \end{aligned} \qquad (*)$$

If the last system (*) is hyperbolic (Cauchy problem well-posed) then the wave is *stable under slow modulations* (Whitham, 1974).

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Equivalently (Whitha, 1965) we may express (*) in terms of E and c. Averaged Lagrangian:

$$\begin{aligned}\langle L\rangle &= \frac{1}{T}\int_0^T \tfrac{1}{2}(c^2-1)f_z(z)^2 - V(f(z))\,dz\\ &= \frac{\sqrt{2}}{T}\oint\left((c^2-1)(E-V(\eta))\right)^{1/2}d\eta - E =: \mathcal{L}(E,c).\end{aligned}$$

$$\mathcal{L}(E,c) = \frac{2\sqrt{2}}{T}\sqrt{c^2-1}\int_{v_1}^{v_2}\sqrt{E-V(\eta)}\,d\eta - E, \qquad \text{(sup, lib)},$$

$$\mathcal{L}(E,c) = -\frac{2\sqrt{2}}{T}\sqrt{1-c^2}\int_{v_3}^{v_4}\sqrt{V(\eta)-E}\,d\eta - E, \qquad \text{(sub, lib)},$$

$$\mathcal{L}(E,c) = \frac{\sqrt{2}}{T}\sqrt{c^2-1}\int_{0}^{P}\sqrt{E-V(\eta)}\,d\eta - E, \qquad \text{(sup, rot)},$$

$$\mathcal{L}(E,c) = -\frac{\sqrt{2}}{T}\sqrt{1-c^2}\int_{0}^{P}\sqrt{V(\eta)-E}\,d\eta - E, \qquad \text{(sub, rot)}.$$

Define:

$$W(E,c) = \sqrt{2}\oint \left((c^2-1)(E-V(\eta))\right)^{1/2} d\eta,$$

$$W(E,c) := \operatorname{sgn}(c^2-1)\sqrt{|c^2-1|}J(E),$$

$$J(E) := \begin{cases} J_L(E), & \text{librations,} \\ J_R(E), & \text{rotations,} \end{cases}$$

$$J_R(E) := \sqrt{2}\int_0^P \sqrt{\operatorname{sgn}(c^2-1)(E-V(\eta))}\, d\eta$$

$$J_L(E) := 2\sqrt{2}\int_{v_i}^{v_j} \sqrt{\operatorname{sgn}(c^2-1)(E-V(\eta))}\, d\eta$$

Lemma

For each of the four cases under consideration (sub- or superluminal, libration or rotation) there hold

$$W_E = T, \tag{1}$$

$$W_c = \frac{cW}{c^2 - 1}. \tag{2}$$

Taking average of conservation of energy and momentum equations we can express the Whitham modulation system (*) as:

$$\begin{aligned} \left(\frac{W_c}{T}\right)_t + \left(\frac{cW_c}{T} - E\right)_x &= 0, \\ \left(\frac{1}{T}\right)_t + \left(\frac{c}{T}\right)_x &= 0. \end{aligned} \qquad (**)$$

Lemma

Whitham's system of equations (**) *is equivalent to the system:*

$$\begin{pmatrix} E \\ c \end{pmatrix}_t + A(E,c) \begin{pmatrix} E \\ c \end{pmatrix}_x = 0, \qquad \text{(Wh)}$$

$$A(E,c) = \frac{1}{N(E,c)} \begin{pmatrix} c(J(E)J''(E)+J'(E)^2) & -J(E)J'(E) \\ (c^2-1)^2 J'(E)J''(E) & c(J(E)J''(E)+J'(E)^2) \end{pmatrix},$$

$$N(E,c) = J(E)J''(E) + c^2 J'(E)^2.$$

Lemma

Whitham system (Wh) *is hyperbolic if and only if*

$$J''(E) < 0.$$

Characteristic velocities:

$$c(J(E)J''(E) + J'(E)^2) - s_{\pm} = \pm|c^2 - 1|\big(-J(E)J''(E)J'(E)^2\big)^{1/2}.$$

Proof of Whitham's modulational instability

Lemma

$$\operatorname{sgn} J''(E) = -\rho.$$

Proof:

$$T_E = W_{EE} = \operatorname{sgn}(c^2 - 1)\sqrt{|c^2 - 1|}J''(E).$$

Corollary

The quasilinear Whitham system (Wh) *is hyperbolic if and only if* $\rho = 1$*. In this case we say that the underlying periodic traveling wave is modulationally stable (otherwise we say it is modulationally unstable).*

Theorem (Proof of Whitham's instability)

Under the non-degenerate condition $T_E \neq 0$*, if the periodic traveling wave is modulationaly unstable in the sense defined by Whitham then it is spectrally unstable.*

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Under the non-degenerate condition $T_E \neq 0$*, if the periodic traveling wave is modulationaly unstable in the sense defined by Whitham then it is spectrally unstable.*

Corollary

Under the non-degenerate condition $T_E \neq 0$, a necessary condition for the spectral stability of a periodic wave is that the modulational instabilty index is $\rho = 1$, or equivalently, that the Whitham modulation system is hyperbolic.

Finally we recover:

Theorem (Whitham, 1974)

- *Both super- and subluminal rotational waves are modulationally stable,*
- *Both super- and subluminal librational waves are modulationally unstable (and whence, spectrally unstable).*

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(In)stability in the rotational case

Theorem

Under assumptions we have:

(A) *Superluminal rotational waves are spectrally unstable.*

(B) *Subluminal rotational waves are spectrally stable.*
That is: if $\lambda \in \sigma$ then λ is purely imaginary.

Part (A):
Define $G : \mathbb{C} \to \mathbb{R}$ by

$$G(\lambda) = \log|\mu_+(\lambda)| \log|\mu_-(\lambda)|.$$

G continuous in $\mathbb{R}^2$ and $\lambda \in \sigma$ if and only if $G(\lambda) = 0$. Fact: if $\mu(\lambda) \in \sigma\mathbb{M}(\lambda)$ (Floquet mult. for (P) then $\eta(\lambda) = \exp(-\lambda cT/(c^2-1)) \in \sigma\mathbb{M}_H(\lambda)$ (Floquet mult. for (H)). By Abel's identity:

$$\begin{aligned} G(\lambda) &= \left(\text{Re}\, \frac{c\lambda T}{c^2-1}\right)^2 - (\log|\eta_+(\lambda)|)^2 \\ &= \left(\text{Re}\, \frac{c\lambda T}{c^2-1}\right)^2 - (\log|\eta_-(\lambda)|)^2. \end{aligned}$$

Thus, for $\lambda \in i\mathbb{R}$, $G \leq 0$. Moreover, $G(i\beta) = 0$ iff $i\beta \in \sigma \cap i\mathbb{R} = \sigma^H \cap i\mathbb{R}$. Thus,

Corollary

Suppose $\beta \in \mathbb{R}$ *is such that* $\left(\frac{i\beta}{c^2-1}\right)^2 \notin \sigma^H$. *Then* $G(i\beta) < 0$.

Moreover, we can show:

Lemma

For a superluminal rotational wave, $G(\lambda) > 0$ for $\lambda \in \mathbb{R}$, $\lambda \gg 1$, and there is a $i\beta_*$ in the spectral gap of σ_H, that is, $G(i\beta) < 0$.

By continuity, there must be an eigenvalue $\lambda = \alpha_* t + i\beta_*(1-t)$ for some $t \in (0,1)$, where $G(\alpha_*) > 0$, α_* large and real, such that $G(\lambda) = 0$. Clearly, Re $\lambda > 0$.

This shows (A).

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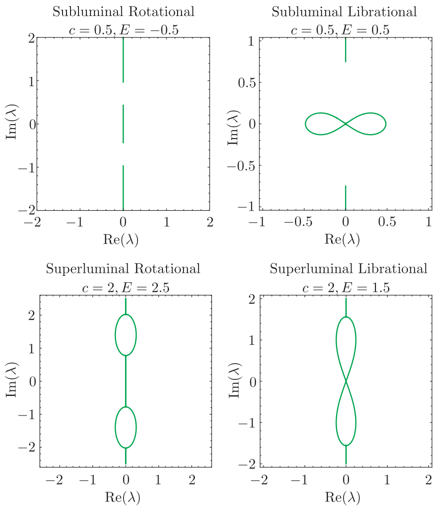

Figure: Numerical plots of the Floquet spectrum $G(\lambda) = 0$ for sine-Gordon.

Part (B): Spectral stability of subluminal rotations.

By energy estimates: define the Hamiltonian operator $H = d^2/dz^2 + V''(f)/(c^2 - 1)$ so that the spectral equation (P) is:

$$(c^2 - 1)Hw(z) - 2c\lambda w_z(z) + \lambda^2 w(z) = 0$$

Lemma

The operator H is negative semidefinite in the case of a rotational wave. For librations, H is indefinite.

If $\lambda \in \sigma$, multiply eq. by w^* and integrate by parts on a fundamental period $[0,T]$:

$$(c^2-1)\langle w, Hw\rangle - 2im\lambda + \|w\|^2\lambda^2 = 0,$$

$$m := -ic\int_0^T w(z)^* w_z(z)\,dz \in \mathbb{R}$$

$m \in \mathbb{R}$ using the periodicity of w and integrating by parts. The roots of the quadratic are:

$$\lambda = \frac{1}{\|w\|^2}\left[im \pm \sqrt{-m^2-(c^2-1)\|w\|^2\langle w, Hw\rangle}\right].$$

$\lambda \in i\mathbb{R}$ whenever $c^2 < 1$. This shows (B).

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Thank you!