On the spectral, modulational and orbital stability of periodic wavetrains for the sine-Gordon equation

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1. Introduction
2. Analysis of the monodromy map
3. Spectral (in)stability results
4. Multidimensional orbital stability

The nonlinear Klein-Gordon equation

Nonlinear Klein-Gordon with periodic potential:

$$u_{tt} - u_{xx} + V'(u) = 0. \qquad \text{(nKG)}$$

for $(x,t) \in \mathbb{R} \times [0, +\infty)$, u scalar, $V \in C^2$, periodic.

Sine-Gordon equation:

$$u_{tt} - u_{xx} + \sin u = 0. \qquad \text{(SG)}$$

$$V(u) = 1 - \cos u$$

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Applications (sine-Gordon):

- Surfaces with negative Gaussian curvature (Eisenhart, 1909)
- Propagation of crystal dislocations (Frenkel and Kontorova, 1939)
- Elementary particles (Perring and Skyrme, 1962)
- Propagation of magnetic flux on a Josephson line (Scott, 1969)
- Dynamics of fermions in the Thirring model (Coleman, 1975)
- Oscillations of a rigid pendulum attached to a stretched rubber band (Drazin, 1983)

Assumptions on the potential:

(a) $V : \mathbb{R} \to \mathbb{R}$ is of class C^2 in all its domain and it is periodic with fundamental period P.

(b) V has only non-degenerate critical points.

(c) $V'(u)^4 (V(u)/V'(u)^2)'' \geq 0$ for all u under consideration.

Assumption (c) implies monotonicity of the period map with respect to the energy.

Traveling waves

$u(x,t) = f(x-ct)$, $z = x - ct$, solution to the nonlinear pendulum equation:

$$(c^2-1)f_{zz} + V'(f(z)) = 0,$$

Sine-Gordon case:

$$(c^2-1)f_{zz} + \sin(f(z)) = 0,$$

$c \in \mathbb{R}$ (wave speed), $c^2 \neq 1$.

Upon integration:

$$\frac{1}{2}(c^2-1)f_z^2 = E - V(f),$$

E = constant (energy). Under assumptions:
$0 < E_0 = \max V(u)$

Sine-Gordon case: $V(u) = 1 - \cos u$, $E_0 = 2$,

$$\frac{1}{2}(c^2-1)f_z^2 = E - 1 + \cos f(z).$$

W.l.o.g.

(d) V has fundamental period $P = 2\pi$ and

$$\min_{u\in\mathbb{R}} V(u) = 0, \qquad \max_{u\in\mathbb{R}} V(u) = 2.$$

Classification

First dichotomy (wave speed):

- **Subluminal** waves: $c^2 < 1$
- **Superluminal** waves: $c^2 > 1$

Second dichotomy (energy E):

- **Librational** wavetrain: $f(z+T) = f(z)$. Closed trajectory inside the separatrix in the phase portrait.
- **Rotational** wavetrain: $f(z+T) = f(z) \pm 2\pi$. Open trajectory outside the separatrix in the phase plane. Sign f_z is fixed.

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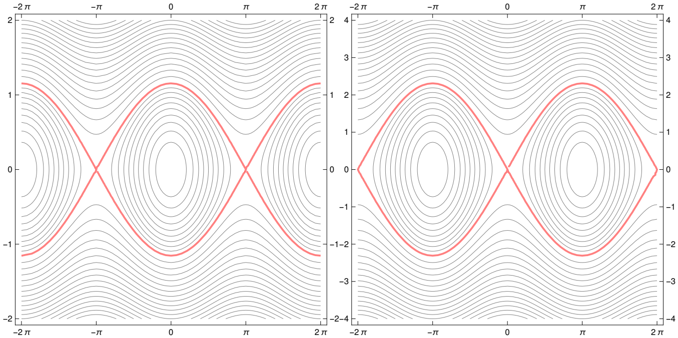

Figure : Phase portrait sine-Gordon case: $V(u) = 1 - \cos u$: superluminal $c^2 > 1$ (left); subluminal $c^2 < 1$ (right).

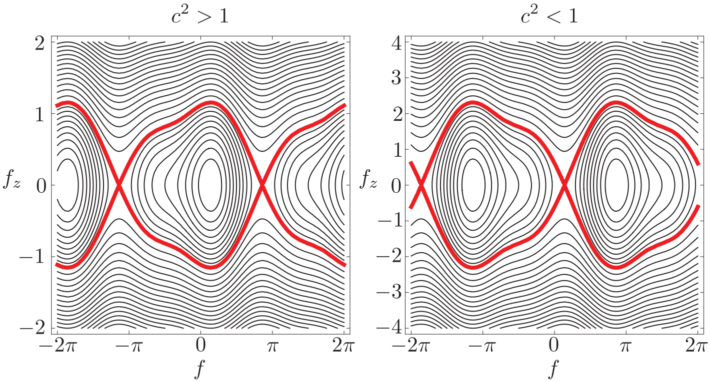

Figure : Phase portrait for $V(u) = 1 - (0.861)(\cos u + \frac{1}{3}\sin(2u))$: superluminal $c^2 > 1$ (left); subluminal $c^2 < 1$ (right).

Superluminal librational: $c^2 > 1$, $0 < E < E_0$.
$\mathcal{K}(E) = \{u \in \mathbb{R} : (E - V(u))/(c^2 - 1) \geq 0\}$ = disjoint union of intervals in $(0, \pi)$. In (v_1, v_2), only one non-degenerate zero of V'. Librational (closed) periodic orbit.

$$f_z = \frac{\sqrt{2}}{\sqrt{c^2 - 1}} \sqrt{E - V(f)},$$

where $f \in (v_1, v_2) \subset \mathcal{K}(E)$.

$$T = \sqrt{2}\sqrt{c^2 - 1} \int_{v_1}^{v_2} \frac{d\eta}{\sqrt{E - V(\eta)}}.$$

Sine-Gordon: wave oscillates around $f = 0$, in $(v_1, v_2) = (-\mathrm{Arc}\cos(-E+1), \mathrm{Arc}\cos(-E+1))$

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$\mathcal{K}(E) = \{u \in \mathbb{R} : (V(u) - E)/(1-c^2) \geq 0\}$ = disjoint union of intervals in $(0,\pi)$. In (v_3, v_4), only one non-degenerate zero of V'. Librational (closed) periodic orbit.

$$f_z = \frac{\sqrt{2}}{\sqrt{1-c^2}}\sqrt{V(f)-E},$$

where $f \in (v_3, v_4) \subset \mathcal{K}(E)$.

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Superluminal rotational: $c^2 > 1$, $E > E_0$, $E - V(f) > 0$ and $\mathcal{K}(E) = \mathbb{R}$. Rotation, f_z has fixed sign. Orbit outside the separatrix and $f(z+T) = f(z) \pm \pi$ for all z.

$$f_z^2 = \frac{2(E - V(f))}{c^2 - 1} > 0,$$

$$T = \frac{\sqrt{c^2-1}}{\sqrt{2}} \int_0^{\pi} \frac{d\eta}{\sqrt{E - V(\eta)}}$$

Subluminal rotational: $c^2 < 1$, $E < 0$, $V(f) - E \geq 0$ and $\mathcal{K}(E) = \mathbb{R}$ with . f_z has fixed sign. Orbit outside the separatrix and $f(z+T) = f(z) \pm \pi$ for all z.

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$$\mathbb{G}^{\text{lib}}_{<} = \{c^2 < 1,\ 0 < E < E_0\},\ \text{(subluminal librational)},$$
$$\mathbb{G}^{\text{rot}}_{<} = \{c^2 < 1,\ E < 0\},\ \text{(subluminal rotational)},$$
$$\mathbb{G}^{\text{lib}}_{>} = \{c^2 > 1,\ 0 < E < E_0\},\ \text{(superluminal librational)},$$
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$$(E,c) \in \mathbb{G} := \mathbb{G}^{\text{lib}}_{<} \cup \mathbb{G}^{\text{rot}}_{<} \cup \mathbb{G}^{\text{lib}}_{>} \cup \mathbb{G}^{\text{rot}}_{>}$$

Lemma

For each fixed $z \in \mathbb{R}$, $f(z; E, c)$ *is of class* C^2 *in* $(E,c) \in \mathbb{G}$.

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Spectral problem

Solution $f(z)+u(z,t)$, with $u=$ perturbation:

$$u_{tt}-2cu_{zt}+(c^2-1)u_{zz}+V'(u+f)-V'(f)=0.$$

Linearized equation:

$$u_{tt}-2cu_{zt}+(c^2-1)u_{zz}+V''(f(z))u=0.$$

$u=w(z)e^{\lambda t}$, $\lambda\in\mathbb{C}$, $w\in X$ Banach:

$$(c^2-1)w_{zz}-2c\lambda w_z+(\lambda^2+V''(f(z)))w=0. \qquad \text{(P)}$$

Quadratic "pencil" in λ.

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Quadratic “pencil” in λ.

Floquet spectrum

Formally: $\lambda \in \sigma_F$ is a Floquet eigenvalue if there exists a bounded solution w to (P). (Precise definition in a moment.)

We say the wave is *spectrally stable* if $\sigma_F \subset \{\text{Re}\, \lambda \leq 0\}$. Otherwise it is *spectrally unstable*.

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Previous results

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Summary of stability results

Wave	Whitham (1974)	Scott (1969)
Subluminal rotational	stable	stable
Superluminal rotational	stable	unstable
Subluminal librational	unstable	unstable
Superluminal librational	unstable	unstable

Whitham (1965, 1974):
Modulation theory: well established (formal) physical method based on WKB expansions. Exact wave $f = f(x-ct) = \tilde{f}(kx - \omega t)$. Allowing dependence $k = k(x,t), \omega = \omega(x,t)$, under "slow modulations", if the PDE system on (k, ω) is well-posed then the wave is "stable".

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Scott (1969):

$$y = \exp\left(\frac{-c\lambda z}{c^2-1}\right),$$

$$y_{zz} + \frac{V''(f(z))}{c^2-1} y = \left(\frac{\lambda}{c^2-1}\right)^2 y =: \nu y. \qquad \text{(H)}$$

Hill's equation with period T. $\nu \in \sigma_H$ (Floquet spectrum of (H)) if there is a bounded solution y.

Scott assumed that the transformation is *isospectral*: ($\sigma_H = \sigma_F$). This is **not true**. Actually:

Lemma (JMMP1)

If $\lambda \in \sigma_H \cap \sigma_F$ then $\lambda \in i\mathbb{R}$.

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References:

- C.K.R.T. Jones, R. Marangell, P.D. Miller, R.P., *On the stability of periodic traveling sine-Gordon waves*, Phys. D **251** (2013) **(JMMP1)**.
- C.K.R.T. Jones, R. Marangell, P.D. Miller, R.P., *Spectral and modulational stability of periodic wavetrains for the nonlinear Klein-Gordon equation.* J. Differential Equations **257** (2014) **(JMMP2)**.
- J. Angulo-Pava, R.P., *Nonlinear orbital stability of subluminal periodic sine-Gordon wavetrains of rotational type.* Preprint (2015) **(AP)**.

Summary:

JMMP1:

- Correct proof of Scott's results (spectral)

JMMP2:

- More generic potentials
- Analysis of the monodromy map
- Modulational stability index
- Relation to Whitham's modulation theory

AP:

- Orbital (nonlinear) stability of subluminal rotational waves
- Multidimensional orbital stability (e.g. 2d sine-Gordon)

Other references (Whitham vs. spectral):

- Serre (2005); Oh, Zumbrun (2006) (viscous conserv. laws)
- Bronski, Johnson (2009); Johnson, Zumbrun (2010); Bronski, Johnson, Zumbrun (2010) (gKdV)
- Johnson (2010) (BBM)
- Noble, Rodrigues (2013) (Kuramoto-Sivashinski)
- Benzoni, Noble, Rodrigues (2013) (Hamiltonian PDEs)

1. Introduction
2. **Analysis of the monodromy map**
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Floquet spectrum

Problem (P) (quadratic pencil) can be written as a first order system:

$$\mathbf{w}_z = \mathbf{A}(z,\lambda)\mathbf{w},$$

$$\mathbf{w} := \begin{pmatrix} w \\ w_z \end{pmatrix},$$

$$\mathbf{A}(z,\lambda) := \begin{pmatrix} 0 & 1 \\ -\dfrac{(\lambda^2 + V''(f(z)))}{c^2-1} & \dfrac{2c\lambda}{c^2-1} \end{pmatrix}.$$

Family of closed, densely defined operators:

$$\mathcal{T}(\lambda) : \mathcal{D} \subset X \to X$$

$$\mathcal{T}(\lambda)W := W_z - \mathbf{A}(z,\lambda)W.$$

E.g.:

$$\mathcal{D} = H^1(\mathbb{R};\mathbb{C}^2), \quad X = L^2(\mathbb{R};\mathbb{C}^2),$$

Spectral stability of periodic waves with respect to *localized perturbations*.

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Spectral stability of periodic waves with respect to *localized perturbations*.

Definition (cf. Sandstede (2002))

The *resolvent* ρ, the *point spectrum* σ_{pt} and the *essential spectrum* σ_{ess} of problem (P):

$$\begin{aligned} \rho &:= \{\lambda \in \mathbb{C} : \ \mathcal{T}(\lambda) \text{ is one-to-one and onto, and} \\ &\quad \mathcal{T}(\lambda)^{-1} \text{ is bounded}\}, \\ \sigma_{\text{pt}} &:= \{\lambda \in \mathbb{C} : \ \mathcal{T}(\lambda) \text{ is Fredholm with zero index} \\ &\quad \text{and has a non-trivial kernel}\}, \\ \sigma_{\text{ess}} &:= \{\lambda \in \mathbb{C} : \ \mathcal{T}(\lambda) \text{ is either not Fredholm or} \\ &\quad \text{has index different from zero}\}. \end{aligned}$$

The *spectrum* is $\sigma = \sigma_{\text{ess}} \cup \sigma_{\text{pt}}$. ($\mathcal{T}(\lambda)$ closed $\Rightarrow \rho = \mathbb{C} \backslash \sigma$.)

Comments:

- The transformation $v_1 = w$, $v_2 = \lambda w$ defines a cartesian product in $X = L^2$ which allows to write as a standard eigenvale problem:

$$\lambda \mathbf{v} = \begin{pmatrix} 0 & 1 \\ -(c^2-1)\partial_z^2 - \cos f(z) & 2c\partial_z \end{pmatrix} \mathbf{v} =: \mathcal{L}\mathbf{v}, \;\; \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

- Equivalent definition to the standard one (essential spectrum of Weyl, 1910): 1-to-1 correspondence between Jordan chains

Lemma (Gardner, 1997)

Since $X = L^2$ all spectrum of problem (P) *is "continuous", that is, $\sigma = \sigma_{ess}$ and σ_{pt} is empty.*

Since (nKG) is a real Hamiltonian system:

Lemma

σ is symmetric with respect to reflection in real and imaginary axes: $\lambda \in \sigma \Rightarrow -\lambda, \bar{\lambda} \in \sigma$.

Spectral stability is equivalent to $\sigma \subset i\mathbb{R}$.

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Monodromy matrix

$$\mathbf{M}(\lambda) := \mathbf{F}(T, \lambda)$$

$\mathbf{F}(z, \lambda) =$ fundamental solution with $\mathbf{F}(0, \lambda) = \mathbf{I}$.

$$\mathbf{M}(\lambda)\mathbf{F}(z, \lambda) = \mathbf{F}(z + T, \lambda)$$

Important feature: $\mathbf{A}$ entire in λ, Picard iterates converge for $\mathbf{F}$ in z bounded $\Rightarrow$ $\mathbf{M}$ is an **entire** (analytic) function of $\lambda \in \mathbb{C}$.

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Floquet multipliers:

$\lambda \in \sigma$ if and only if there exists at least one $\mu \in \mathbb{C}$ (Floquet multiplier) with $|\mu| = 1$ such that

$$\hat{D}(\lambda,\mu) := \det(\mathbf{M}(\lambda) - \mu \mathbf{I}) = 0.$$

$\mu = \mu(\lambda) = e^{i\theta(\lambda)}$ are the eigenvalues of $\mathbf{M}(\lambda)$. $\theta = \theta(\lambda)$ are called the Floquet exponents.

Periodic Evans function

Definition (Gardner, 1997)

The *periodic Evans function* $D : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is

$$D(\lambda, \kappa) := \hat{D}(\lambda, e^{i\kappa T}) = \det(\mathbf{M}(\lambda) - e^{i\kappa T}\mathbf{I}),$$

for each $(\lambda, \kappa) \in \mathbb{C} \times \mathbb{R}$.

Properties: (Gardner 1997, 1998)

- σ is the set of all $\lambda \in \mathbb{C}$ such that $D(\lambda, \kappa) = 0$ for some real κ.
- D is analytic in λ and κ.
- The order of the zero in λ is the multiplicity of the eigenvalue.
- $\hat{D}(\lambda, 1) = D(\lambda, 0)$ detects spectra corresponding to perturbations which are T-periodic.

Floquet spectrum

Boundary value problem of the form

$$(c^2-1)w_{zz}-2c\lambda w_z+(\lambda^2+V''(f(z)))w=0,$$

$$\begin{pmatrix} w(T) \\ w_z(T) \end{pmatrix} = e^{i\theta} \begin{pmatrix} w(0) \\ w_z(0) \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

For a given $\theta \in \mathbb{R}$ we define $\sigma_\theta \subset \mathbb{C}$ to be the set of complex λ for which there exists a nontrivial solution. The Floquet spectrum σ_F is defined then as the union over θ of these sets:

$$\sigma_F := \bigcup_{-\pi<\theta\leq\pi} \sigma_\theta.$$

Observations:

- Clearly $\sigma = \sigma_F$
- Each set σ_θ is discrete: zero set of the entire function $\det(\mathbf{M}(\lambda) - e^{i\theta} I)$
- The set σ_0 (with $\theta = 0$) is the part of the spectrum corresponding to perturbations which are co-periodic (*periodic partial spectrum*)
- θ - local coordinate; *curves of spectrum*: if $D_\lambda(\lambda_0, \mu_0) \neq 0$, $D_\mu(\lambda_0, \mu_0) \neq 0$ then σ is a smooth local curve
- At points where derivatives vanish: spectral analytic arcs (e.g. at $\lambda = 0$!)

Bloch wave decomposition

Transformation: $\mathbf{y} = e^{-i\theta z/T}\mathbf{w}$ yields

$$\mathbf{y}_z = \widetilde{\mathbf{A}}(z,\lambda,\theta)\mathbf{y},$$

$$\widetilde{\mathbf{A}}(z,\lambda,\theta) = \mathbf{A}(z,\lambda) - (i\theta/T)\mathbf{I},$$

Boundary conditions:

$$\mathbf{y}(0) = \mathbf{y}(T)$$

$\lambda \in \sigma$ iff for some $-\pi < \theta \leq \pi$ there exists a non-trivial solution $\mathbf{y} \in H^1_{\text{per}}([0,T];\mathbb{C}^2)$

Bloch wave decomposition

Transformation: $\mathbf{y} = e^{-i\theta z/T}\mathbf{w}$ yields

$$\mathbf{y}_z = \widetilde{\mathbf{A}}(z,\lambda,\theta)\mathbf{y},$$

$$\widetilde{\mathbf{A}}(z,\lambda,\theta) = \mathbf{A}(z,\lambda) - (i\theta/T)\mathbf{I},$$

Boundary conditions:

$$\mathbf{y}(0) = \mathbf{y}(T)$$

$\lambda \in \sigma$ iff for some $-\pi < \theta \leq \pi$ there exists a non-trivial solution $\mathbf{y} \in H^1_{\mathrm{per}}([0,T];\mathbb{C}^2)$

Equivalently, $w = e^{1\theta z}q$ transforms the spectral pencil (P) into

$$\Big((c^2-1)\big(\partial_z + \frac{i\theta}{T}\big)^2 - 2c\lambda\big(\partial_z + \frac{i\theta}{T}\big) + (\lambda^2 + \cos f(z))\Big)q = 0,$$

$$q(T) = q(0), \quad q_z(T) = q_z(0)$$

Scalar domain base space $H^1_{\text{per}}([0,T];\mathbb{C}) \subset L^2_{\text{per}}([0,T];\mathbb{C})$. Make $p_1 = q$, $p_2 = \lambda q$, we obtain:

Family of Bloch operators

$$\lambda \mathbf{p} = \begin{pmatrix} 0 & 1 \\ -(c^2-1)\big(\partial_z + \frac{i\theta}{T}\big)^2 - \cos f(z) & 2c\big(\partial_z + \frac{i\theta}{T}\big) \end{pmatrix} \mathbf{p} =: \mathcal{L}_\theta \mathbf{p},$$

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

$$\mathcal{L}_\theta : \mathcal{D} = H^1_{\mathrm{per}}([0,T];\mathbb{C}^2) \to L^2_{\mathrm{per}}([0,T];\mathbb{C}^2),$$

$\lambda \in \sigma$ (continuous) iff $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L}_\theta)$ for some $\theta \in (-\pi, +\pi]$

Solutions at $\lambda = 0$

$$f = f(z; E, c), \ (E, c) \in \mathbb{G}.$$

w solution to pencil (P), with initial conditions:

$$\begin{aligned} w(0;E,c) &= f(0;E,c) \\ &= \begin{cases} f(T;E,c), & E \in (0, E_0), \quad \text{(lib)}, \\ f(T;E,c) - \pi, & E \in (-\infty, 0) \cup (E_0, +\infty), \quad \text{(rot)}, \end{cases} \end{aligned}$$

$$\partial_z w(0;E,c) = f_z(0;E,c) = f_z(T;E,c)$$

System at $\lambda = 0$:

$$\mathbf{w}_z = \mathbf{A}(z,0)\mathbf{w},$$

$$\mathbf{A}(z,0) = \begin{pmatrix} 0 & 1 \\ -V''(f(z))/(c^2-1) & 0 \end{pmatrix}.$$

Lemma

The two-dimensional vector space of solutions is spanned by

$$\mathbf{w}_0(z) = \begin{pmatrix} f_z \\ f_{zz} \end{pmatrix}, \quad \textit{and} \quad \mathbf{w}_1(z) = \begin{pmatrix} f_E \\ f_{Ez} \end{pmatrix}.$$

$$\det(\mathbf{w}_0(z), \mathbf{w}_1(z)) = f_z f_{Ez} - f_E f_{zz} = \frac{1}{c^2-1} \neq 0$$

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$$\det(\mathbf{w}_0(z), \mathbf{w}_1(z)) = f_z f_{Ez} - f_E f_{zz} = \frac{1}{c^2-1} \neq 0$$

Solution matrix:

$$\mathbf{Q}(z,0) := (\mathbf{w}_0(z), \mathbf{w}_1(z))$$

$$\mathbf{F}(z,0) = \mathbf{Q}(z,0)\mathbf{Q}(0,0)^{-1}.$$

$$\mathbf{M}(0) = \mathbf{F}(T,0) = \mathbf{Q}(T,0)\mathbf{Q}(0,0)^{-1}$$

$$\mathbf{Q}(z,0)^{-1} = (c^2-1)\begin{pmatrix} f_{Ez} & -f_E \\ -f_{zz} & f_z \end{pmatrix}.$$

Lemma

If $T_E \neq 0$, there exists a basis in $\mathbb{R}^2$ such that the monodromy map $\mathbf{M}(\lambda)$ at $\lambda = 0$ has the Jordan form

$$\mathbf{M}(0) \sim \begin{pmatrix} 1 & -T_E \\ 0 & 1 \end{pmatrix}.$$

$\mathbf{Q}(T,0) - \mathbf{Q}(0,0)$ is a rank-one matrix provided that $T_E \neq 0$:

$$\mathbf{Q}(T,0) = \mathbf{Q}(0,0) + \begin{pmatrix} 0 & -T_E v_0(E,c) \\ 0 & -T_E \frac{V'(u_0(E,c))}{c^2-1} \end{pmatrix}$$

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Under Assumption (c), we have monotonicity of the period map (Chicone, 1987: criterion for planar Hamiltonian systems):

Lemma

Under assumptions there holds $T_E \neq 0$. *More precisely we have:*

(i) $T_E > 0$ *in the rotational subluminal and librational superluminal cases.*

(ii) $T_E < 0$ *in the rotational superluminal and librational subluminal cases.*

Lemma

If we define

$$\bar{\Delta} := -\frac{T_E}{c^2 - 1}$$

then

(a) $\bar{\Delta} > 0$ *for rotational waves.*

(b) $\bar{\Delta} < 0$ *for librational waves.*

Solutions series expansions

$\mathbf{Q} = \mathbf{Q}(z,\lambda)$ solution to

$$\frac{d\mathbf{Q}}{dz} = \mathbf{A}(z,\lambda)\mathbf{Q}.$$

$$\mathbf{Q}(0,\lambda) = \mathbf{Q}(0,0) = (\mathbf{w}_0(0),\ \mathbf{w}_1(0))$$

By analyticity, seek series expansion

$$\mathbf{Q}(z,\lambda) = \sum_{n=0}^{+\infty} \lambda^n \mathbf{Q}_n(z)$$

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Collecting like powers of λ we obtain a hierarchy:

$$(c^2-1)\frac{d\mathbf{Q}_1}{dz} = \mathbf{A}_0(z)\mathbf{Q}_1 + \mathbf{A}_1\mathbf{Q}_0$$

$$(c^2-1)\frac{d\mathbf{Q}_n}{dz} = \mathbf{A}_0(z)\mathbf{Q}_n + \mathbf{A}_1\mathbf{Q}_{n-1} + \mathbf{A}_2\mathbf{Q}_{n-2}, \;\; n = 2, 3, \ldots$$

Solution by variation of parameters:

$$\mathbf{Q}_1(z) = \frac{\mathbf{Q}_0(z)}{c^2-1}\int_0^z \mathbf{Q}_0(y)^{-1}\mathbf{A}_1\mathbf{Q}_0(y)\,dy$$

$$\mathbf{Q}_n(z) = \frac{\mathbf{Q}_0(z)}{c^2-1}\int_0^z \mathbf{Q}_0(y)^{-1}\big(\mathbf{A}_1\mathbf{Q}_{n-1}(y) + \mathbf{A}_2\mathbf{Q}_{n-2}\big)\,dy, \;\; n \geq 2$$

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By Abel's identity:

Lemma

For all $z \in \mathbb{R}$, $\lambda \in \mathbb{C}$, *there holds*

$$\det \mathbf{Q}(z,\lambda) = \frac{\exp\left(2c\lambda z/(c^2-1)\right)}{c^2-1}.$$

After (tedious) computations:

Lemma

$$\operatorname{tr} \mathbf{Q}_0(T)\mathbf{Q}_0(0)^{-1} = 2.$$

$$\operatorname{tr} \mathbf{Q}_1(T)\mathbf{Q}_0(0)^{-1} = \frac{2cT}{c^2-1}.$$

$$\operatorname{tr} \mathbf{Q}_2(T)\mathbf{Q}_0(0)^{-1} = \frac{c^2T^2}{(c^2-1)^2} - \frac{T_E}{c^2-1}\int_0^T f_z(y)^2\,dy.$$

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Perturbation of the Jordan block

By analyticity of the monodromy map:

$$\mathbf{M}(\lambda) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} \frac{d^n \mathbf{M}}{d\lambda^n}(0).$$

(Standard perturbation theory, Kato.) In general, the Floquet multipliers bifurcate from $\lambda = 0$ in Pusieux series.

Fundamental matrix:

$$\mathbf{F}(z,\lambda) = \mathbf{Q}(z,\lambda)\mathbf{Q}_0(0)^{-1} = \sum_{n=0}^{+\infty} \lambda^n \mathbf{Q}_n(z)\mathbf{Q}_0^{-1} =: \sum_{n=0}^{+\infty} \lambda^n \mathbf{F}_n(z)$$

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Lemma

We have convergent series expansions

$$\mathbf{M}(\lambda) = \sum_{n=0}^{+\infty} \lambda^n \mathbf{Q}_n(T)\mathbf{Q}_0(0)^{-1},$$

$$\operatorname{tr}\mathbf{M}(\lambda) = \sum_{n=0}^{+\infty} \lambda^n \operatorname{tr}\mathbf{Q}_n(T)\mathbf{Q}_0(0)^{-1},$$

and $$\det\mathbf{M}(\lambda) = \sum_{n=0}^{+\infty} \left(\frac{2cT}{c^2-1}\right)^n \frac{\lambda^n}{n!},$$

Expansion of the Floquet multipliers

μ, solutions to:

$$\hat{D}(\lambda,\mu) = \det(\mathbf{M}(\lambda) - \mu\mathbf{I}) = \mu^2 - (\operatorname{tr}\mathbf{M}(\lambda))\mu + \det\mathbf{M}(\lambda) = 0$$

$$\mu_{\pm}(\lambda) = \frac{1}{2}\Big(\operatorname{tr}\mathbf{M}(\lambda) \pm \big((\operatorname{tr}\mathbf{M}(\lambda))^2 - 4\det\mathbf{M}(\lambda)\big)^{1/2}\Big)$$

Expanding:

$$
\begin{aligned}
&\operatorname{tr}\mathbf{M}(\lambda)^2 - 4\det\mathbf{M}(\lambda) = \\
&\Big(\operatorname{tr}\mathbf{Q}_0(T)\mathbf{Q}_0(0)^{-1} + \lambda\operatorname{tr}\mathbf{Q}_1(T)\mathbf{Q}_0(0)^{-1} + \lambda^2\operatorname{tr}\mathbf{Q}_2(T)\mathbf{Q}_0(0)^{-1}\Big)^2 + \\
&\quad -4\Big(1 + \frac{2cT}{c^2-1}\lambda + \frac{2c^2T^2}{(c^2-1)^2}\lambda^2\Big) + O(\lambda^3) \\
&= 4\Delta\lambda^2 + O(\lambda^3),
\end{aligned}
$$

where,

$$
\Delta := -\frac{T_E}{c^2-1}\int_0^T f_z(y)^2\,dy
$$

The two Floquet multipliers are analytic functions of λ at $\lambda = 0$. Asymptotic form:

$$\mu_{\pm}(\lambda) = 1 + \Big(\frac{cT}{c^2 - 1} \pm \Delta^{1/2}\Big)\lambda + O(\lambda^2)$$

Definition

We define the *modulational instability index* to be the quantity

$$\gamma_M := \operatorname{sgn} \Delta$$

Clearly $\operatorname{sgn} \Delta = \operatorname{sgn} \bar{\Delta}$

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Expansion of D near the origin

Lemma

The periodic Evans function $D(\lambda,\kappa)$, for $(\lambda,\kappa) \in \mathbb{C}\times\mathbb{R}$, has the following expansion in a neighborhood of $(\lambda,\kappa) = (0,0)$,

$$D(\lambda,\kappa) = -\Delta\lambda^2 + \left(i\kappa - \frac{cT}{c^2-1}\lambda\right)^2 + O(3),$$

where $O(3)$ denotes terms of order three or higher in (λ,k).

Lemma

If $\gamma_M = 1$ then the solutions to $D(\lambda,\kappa) = 0$ near $(\lambda,\kappa) = (0,0)$ emerge from the origin tangentially to the imaginary axis in the complex λ-plane:

$$\lambda(\kappa) = -iv\kappa + O(\kappa^2),$$

with $v \in \mathbb{R}$, for $|\kappa| \ll 1$.
If $\gamma_M = -1$ then the solutions emerge from the origin tangentially to two lines passing through the origin and forming non-zero angles with the imaginary axis:

$$\lambda(\kappa) = -(\alpha + i\beta)\kappa + O(\kappa^2),$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, for $|\kappa| \ll 1$.

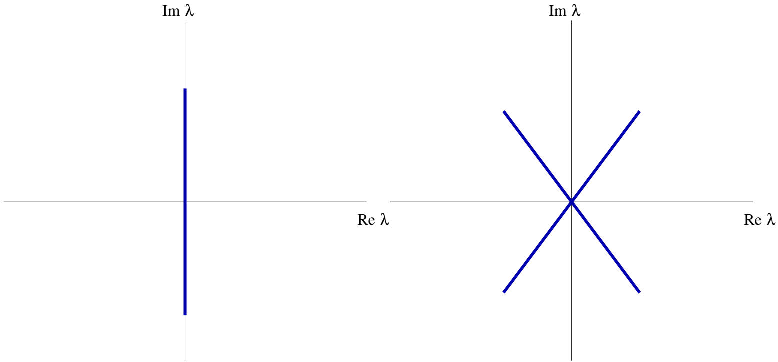

Figure : Qualitative sketch of σ near the origin. $\gamma_M = 1$ (left); $\gamma_M = -1$ (right).

Theorem

Under assumptions (a), (b) and (c):

- $\gamma_M = -1$ *for librational waves. Spectrally unstable.*
- $\gamma_M = 1$ *for rotational waves. The spectrum is tangent to the imaginary axis at* $\lambda = 0$*.*

Theorem

Under the non-degeneracy condition $T_E \neq 0$ *if the modulational instability index is* $\gamma_M = -1$ *then the underlying periodic traveling wave is spectrally unstable.*

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Relation to Whitham's modulation theory

Reference: Whitham, Proc. Roy. Soc. Ser. A (1965).

WKB approximations of the form:

$$u(x,t) = f\left(\frac{z(x,t)}{\varepsilon}\right) + O(\varepsilon),$$

k, ω are no longer constant (and hence, E and c). We have $c = \omega/k$ and $k = \theta_x$, $\omega = -\theta_t$, $\theta = kx - \omega t$. Conservation of fluxons:

$$k_t + \omega_x = 0$$

Averaged Lagrangian

$$I[u] = \int\int L(u, u_x, u_t)\, dx\, dt,$$

$$L(u, u_x, u_t) = \tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_x^2 - V(u).$$

In the wave $u = f(x - ct) = \Phi(kx - \omega t)$:

$$L(u, u_x, u_t) = \tfrac{1}{2}(\omega^2 - k^2)\Phi_\theta(\theta)^2 - V(\Phi(\theta))$$

Averaged Lagrangian:

$$\langle L \rangle = \frac{1}{kT}\int_0^{kT} \tfrac{1}{2}(\omega^2 - k^2)\Phi_\theta(\theta)^2 - V(\Phi(\theta))\, d\theta = \tilde{\mathcal{L}}(\omega, k, E).$$

Averaged Lagrangian variational principle

$$\delta \iint \tilde{\mathcal{L}}(\omega, k, E)\, dx\, dt = 0,$$

$$\tilde{\mathcal{L}}_E = 0, \text{ dispersion relation}$$

Whitham's system:

$$\begin{aligned} k_t + \omega_x &= 0 \\ (\tilde{\mathcal{L}}_\omega)_t - (\tilde{\mathcal{L}}_k)_x &= 0. \end{aligned} \qquad (*)$$

If the last system (*) is hyperbolic (Cauchy problem well-posed) then the wave is *stable under slow modulations* (Whitham, 1974).

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Equivalently (Whitham, 1965) we may express (*) in terms of E and c

$$\begin{aligned}\langle L \rangle &= \frac{1}{T}\int_0^T \tfrac{1}{2}(c^2-1) f_z(z)^2 - V(f(z))\, dz \\ &= \frac{\sqrt{2}}{T} \oint ((c^2-1)(E-V(\eta)))^{1/2}\, d\eta - E =: \mathcal{L}(E,c).\end{aligned}$$

$$\mathcal{L}(E,c) = \frac{2\sqrt{2}}{T}\sqrt{c^2-1}\int_{v_1}^{v_2}\sqrt{E-V(\eta)}\,d\eta - E, \qquad \text{(sup, lib)},$$

$$\mathcal{L}(E,c) = -\frac{2\sqrt{2}}{T}\sqrt{1-c^2}\int_{v_3}^{v_4}\sqrt{V(\eta)-E}\,d\eta - E, \qquad \text{(sub, lib)},$$

$$\mathcal{L}(E,c) = \frac{\sqrt{2}}{T}\sqrt{c^2-1}\int_{0}^{\pi}\sqrt{E-V(\eta)}\,d\eta - E, \qquad \text{(sup, rot)},$$

$$\mathcal{L}(E,c) = -\frac{\sqrt{2}}{T}\sqrt{1-c^2}\int_{0}^{\pi}\sqrt{V(\eta)-E}\,d\eta - E, \qquad \text{(sub, rot)}.$$

Classical action (mechanics):

$$W(E,c) = \sqrt{2}\oint ((c^2-1)(E-V(\eta)))^{1/2}\, d\eta,$$

$$W(E,c) := \operatorname{sgn}(c^2-1)\sqrt{|c^2-1|}J(E),$$

$$J(E) := \begin{cases} J_{\mathrm{lib}}(E), & \text{librations,} \\ J_{\mathrm{rot}}(E), & \text{rotations,} \end{cases}$$

$$J_{\mathrm{rot}}(E) := \sqrt{2}\int_0^P \sqrt{\operatorname{sgn}(c^2-1)(E-V(\eta))}\, d\eta$$

$$J_{\mathrm{lib}}(E) := 2\sqrt{2}\int_{v_i}^{v_j} \sqrt{\operatorname{sgn}(c^2-1)(E-V(\eta))}\, d\eta$$

Lemma

For each of the four cases under consideration (sub- or superluminal, libration or rotation) there hold

$$W_E = T, \tag{1}$$

$$W_c = \frac{cW}{c^2 - 1}. \tag{2}$$

Taking average of conservation of energy and momentum equations we can express the Whitham modulation system (*) as:

$$\left(\frac{W_c}{T}\right)_t + \left(\frac{cW_c}{T} - E\right)_x = 0,$$
$$\left(\frac{1}{T}\right)_t + \left(\frac{c}{T}\right)_x = 0. \qquad (**)$$

Lemma

Whitham's system of equations (**) is equivalent to the system:

$$\begin{pmatrix} E \\ c \end{pmatrix}_t + A(E,c) \begin{pmatrix} E \\ c \end{pmatrix}_x = 0, \qquad \text{(Wh)}$$

$$A(E,c) = \frac{1}{N(E,c)} \begin{pmatrix} c(J(E)J''(E)+J'(E)^2) & -J(E)J'(E) \\ (c^2-1)^2 J'(E)J''(E) & c(J(E)J''(E)+J'(E)^2) \end{pmatrix},$$

$$N(E,c) = J(E)J''(E) + c^2 J'(E)^2.$$

Lemma

Whitham system (Wh) *is hyperbolic if and only if*

$$J''(E) < 0.$$

Characteristic velocities:

$$c(J(E)J''(E) + J'(E)^2) - s_{\pm} = \pm|c^2 - 1|\big(-J(E)J''(E)J'(E)^2\big)^{1/2}.$$

Proof of Whitham's modulational instability

Lemma

$$\operatorname{sgn} J''(E) = -\gamma_M.$$

Proof:

$$T_E = W_{EE} = \operatorname{sgn}(c^2-1)\sqrt{|c^2-1|}J''(E).$$

Corollary

The quasilinear Whitham system (Wh) *is hyperbolic if and only if* $\gamma_M = 1$*. In this case we say that the underlying periodic traveling wave is modulationally stable (otherwise we say it is modulationally unstable).*

Theorem (Proof of Whitham's instability)

Under the non-degenerate condition $T_E \neq 0$*, if the periodic traveling wave is modulationaly unstable in the sense defined by Whitham then it is spectrally unstable.*

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Under the non-degenerate condition $T_E \neq 0$, *if the periodic traveling wave is modulationaly unstable in the sense defined by Whitham then it is spectrally unstable.*

Corollary

Under the non-degenerate condition $T_E \neq 0$, a necessary condition for the spectral stability of a periodic wave is that the modulational instabilty index is $\gamma_M = 1$, or equivalently, that the Whitham modulation system is hyperbolic.

Finally we recover:

Theorem (Whitham, 1974)

- *Both super- and subluminal rotational waves are modulationally stable,*
- *Both super- and subluminal librational waves are modulationally unstable (and whence, spectrally unstable).*

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Interpretation: “Modulational” stability pertains to perturbations for which the wave parameters underlie small variations with respect to wavelength. (Equivalently, perturbations near the origin $\lambda = 0$).

Whitham’s is an *instability theory*.

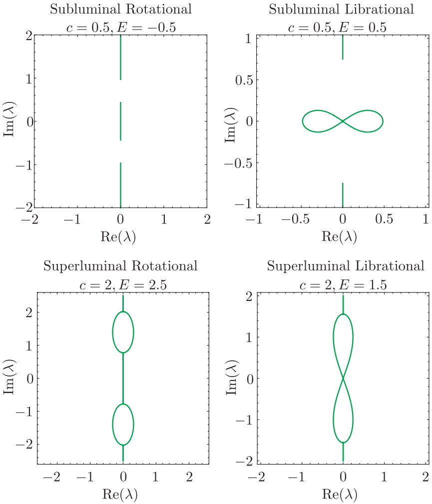

Figure : Numerical plots of the Floquet spectrum $G(\lambda) = 0$ for sine-Gordon.

(In)stability in the rotational case

Theorem

Under assumptions we have:

(A) *Superluminal rotational waves are spectrally unstable.*

(B) *Subluminal rotational waves are spectrally stable. That is: if $\lambda \in \sigma$ then λ is purely imaginary.*

Part (A):
Define $G : \mathbb{C} \to \mathbb{R}$ by

$$G(\lambda) = \log|\mu_+(\lambda)| \log|\mu_-(\lambda)|.$$

G continuous in $\mathbb{R}^2$ and $\lambda \in \sigma$ if and only if $G(\lambda) = 0$. Fact: if $\mu(\lambda) \in \sigma \mathbf{M}(\lambda)$ (Floquet mult. for (P) then $\eta(\lambda) = \exp(-\lambda c T/(c^2 - 1)) \in \sigma \mathbf{M}_H(\lambda)$ (Floquet mult. for (H)). By Abel's identity:

$$\begin{aligned} G(\lambda) &= \left(\text{Re}\, \frac{c\lambda T}{c^2 - 1} \right)^2 - (\log|\eta_+(\lambda)|)^2 \\ &= \left(\text{Re}\, \frac{c\lambda T}{c^2 - 1} \right)^2 - (\log|\eta_-(\lambda)|)^2. \end{aligned}$$

Thus, for $\lambda \in i\mathbb{R}$, $G \leq 0$. Moreover, $G(i\beta) = 0$ iff $i\beta \in \sigma \cap i\mathbb{R} = \sigma^H \cap i\mathbb{R}$. Thus,

Corollary

Suppose $\beta \in \mathbb{R}$ *is such that* $\left(\dfrac{i\beta}{c^2 - 1}\right)^2 \notin \sigma^H$. *Then* $G(i\beta) < 0$.

Moreover, we can show:

Lemma

For a superluminal rotational wave, $G(\lambda) > 0$ for $\lambda \in \mathbb{R}$, $\lambda \gg 1$, and there is a $i\beta_*$ in the spectral gap of σ_H, that is, $G(i\beta) < 0$.

By continuity, there must be an eigenvalue $\lambda = \alpha_* t + i\beta_*(1-t)$ for some $t \in (0,1)$, where $G(\alpha_*) > 0$, α_* large and real, such that $G(\lambda) = 0$. Clearly, Re $\lambda > 0$.

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This shows (A).

Part (B): Spectral stability of subluminal rotations.

By energy estimates: define the Hamiltonian operator $H = d^2/dz^2 + V''(f)/(c^2-1)$ so that the spectral equation (P) is:

$$(c^2-1)Hw(z) - 2c\lambda w_z(z) + \lambda^2 w(z) = 0$$

Lemma

The operator H is negative semidefinite in the case of a rotational wave. For librations, H is indefinite.

If $\lambda \in \sigma$, multiply eq. by w^* and integrate by parts on a fundamental period $[0, T]$:

$$(c^2 - 1)\langle w, Hw\rangle - 2im\lambda + \|w\|^2\lambda^2 = 0,$$

$$m := -ic\int_0^T w(z)^* w_z(z)\, dz \in \mathbb{R}$$

$m \in \mathbb{R}$ using the periodicity of w and integrating by parts. The roots of the quadratic are:

$$\lambda = \frac{1}{\|w\|^2}\left[im \pm \sqrt{-m^2 - (c^2 - 1)\|w\|^2\langle w, Hw\rangle}\right].$$

$\lambda \in i\mathbb{R}$ whenever $c^2 < 1$.

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This shows (B).

Subliminal rotational sine-Gordon wavetrains

$$\begin{cases} \frac{1}{2}(c^2-1)f_z^2 = E + \cos f \\ f(z+T) = f(z) \pm 2\pi, \qquad \text{for all } z \in \mathbb{R}, \\ c^2 < 1 \text{ and } E < -1. \end{cases}$$

Explicit wave form

By quadrature, explicit form of subluminal rotations in terms of elliptic functions:

$$f_{c,E}(z) = \begin{cases} -\arccos^{-1}\left[1-2\,\mathrm{cn}^2\left(\sqrt{\frac{1-E}{2(1-c^2)}}z;k\right)\right], & 0 \leqq z \leqq \frac{T}{2}, \\ \arccos^{-1}\left[1-2\,\mathrm{cn}^2\left(\sqrt{\frac{1-E}{2(1-c^2)}}(T-z);k\right)\right], & \frac{T}{2} \leqq z \leqq T, \end{cases}$$

$$k^2 = \frac{2}{1-E} \in (0,1), \qquad \text{elliptic modulus,}$$

$$\mathrm{cn} = \mathrm{cn}(\cdot), \qquad \text{elliptic cnoidal function}$$

Fundamental period:

$$T = 2\sqrt{\frac{2(1-c^2)}{1-E}}K(k),$$

Complete elliptic integral of the first kind:

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

Properties:

$$f_{c,E}(z+T) = f_{c,E}(z) + 2\pi \quad \text{for all } z \in \mathbb{R}, \quad \text{(rotation)},$$

$$\int_0^T f_{c,E}(z)\,dz = 0, \quad \text{for all } c^2 < 1,\ E < -1, \quad \text{(zero mean)},$$

$$f_{c,E}(z) \to g_c(z) = 4\arctan\big(\exp(z/\sqrt{1-c^2})\big),$$

uniformly on bounded intervals as $E \to -1^-$ (convergence to classical *kink* solution to sine-Gordon).

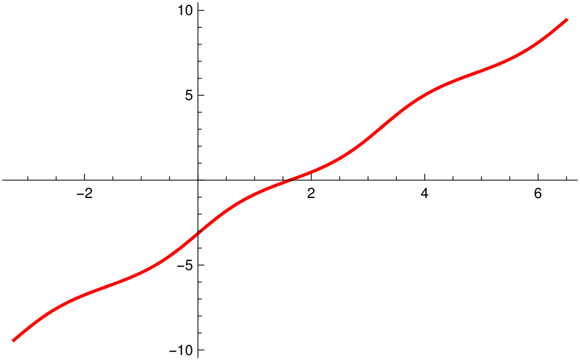

Figure : Rotational subluminal periodic wave $f = f_{c,E}(z)$ with $E = -2$, $c = 0.5$ in the interval $z \in [-T, 2T]$ (plot in red). Here the fundamental period is $T = 3.2476$.

Multidimensional model

Special solution to the **multidimensional sine-Gordon equation**:

$$u_{tt} - \Delta u + \sin u = 0,$$

with $(x,t) \in \mathbb{R}^d \times [0,+\infty)$, $d \geq 2$.

For $d = 2$: models the electrodynamics of extended rectangular Josephson junctions (two layers of superconducting materials separated by an isolating barrier; Josephson, *Adv. in Phys.* (1965))

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Multidimensional spectral stability

Perturbation equations: Seek solutions

$$u(x,y,t) = f(z) + e^{\lambda t} e^{i\xi \cdot y} w(z),$$

where, again, $z = x - ct$, $y \in \mathbb{R}^{d-1}$, $t > 0$.

Linearization, family of quadratic pencils:

$$(c^2 - 1)w_{zz} - 2c\lambda w_z + (\lambda^2 + \cos f(z) + |\xi|^2)w = 0,$$

$\xi \in \mathbb{R}^{d-1}$, $\lambda \in \mathbb{C}$.

Spectrum

$\lambda \in \sigma_\xi$ iff there exists a Floquet multiplier $\mu = e^{i\theta}$, $-\pi < \theta \leqq \pi$ such that there exists a non-trivial solution w to the pencil, with

$$\begin{pmatrix} w(T) \\ w_z(T) \end{pmatrix} = e^{i\theta} \begin{pmatrix} w(0) \\ w_z(0) \end{pmatrix}$$

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Lemma (Multi-d spectral stability)

Let $f = f(z)$ be a subluminal ($c^2 < 1$) rotational ($E < -1$) wave. Then Re $\sigma_\xi = 0$ for all $\xi \in \mathbb{R}$.

Proof: Same!

$$(c^2 - 1)\overline{\mathcal{H}}w(z) - 2c\lambda w_z + (\lambda^2 + \xi^2)w = 0,$$

$$\overline{\mathcal{H}} = \partial_z^2 + \frac{\cos f(z)}{c^2 - 1}, \qquad \text{(Hill's operator)}$$

L^2 energy estimate:

$$(c^2-1)\langle w, \overline{\mathcal{H}w}\rangle - 2im\lambda + (\lambda^2+\xi^2)\|w\|^2 = 0,$$

For $c^2 < 1$, purely imaginary eigenvalues:

$$\lambda = \frac{1}{\|w\|^2}\left(im \pm \sqrt{-(m^2+(c^2-1)\|w\|^2\langle w, \overline{\mathcal{H}w}\rangle + |\xi|^2\|w\|^4)}\right),$$

Well-posedness

Evolution equation for $v(z,y,t) = u(z+ct,y,t) - f(z)$,

$$v_{tt} - 2cv_{zt} + (c^2-1)v_{zz} - v_{yy} + \sin(f(z)+v) - \sin f(z) = 0,$$

i.e.

$$\mathbf{w}_t = J\widetilde{\mathcal{E}}'(\mathbf{w}), \qquad \text{(Ev)}$$

with $\mathbf{w} = (v, v_t)^\top$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 2c\partial_x \end{pmatrix}$, $\widetilde{\mathcal{E}} : H^1_{\mathrm{per}} \times L^2_{\mathrm{per}} \to \mathbb{R}$,

$$\widetilde{\mathcal{E}}(v,w) = \frac{1}{2}\int_0^T \int_0^L (1-c^2)(v_z)^2 + (v_y)^2 + w^2 + 2G(v)\, dy\, dz,$$

$$G'(v(z,y)) = \sin(f(z)+v(z,y)) - \sin f(z)$$

$$Q = \Big\{ g \in L^2_{\rm per}([0,T] \times [0,L_1] \times \cdots \times [0,L_{d-1}]) : \int_0^T g(z,y_1,y_2,\ldots,y_{d-1})dz = 0, \text{ for all } y_i \in [0,L_i] \Big\},$$

$$\mathcal{Z} = (H^1_{\rm per} \times Q) \times Q$$

Theorem

Th initial value problem asociated to equation (Ev) is globally well-posed in $\mathcal{Z}$.

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Orbital stability under multi-d perturbations

Self-adjoint operator:

$$\widetilde{\mathcal{E}}''(0,0) = \begin{pmatrix} (c^2-1)\partial_z^2 - \partial_y^2 + \cos f(z) & 0 \\ 0 & 1 \end{pmatrix}$$

with domain $H^2_{\text{per}} \times L^2_{\text{per}}$.

Stability of $\mathbf{w} = (0,0)^\top$ in $\mathcal{Z}$ under the flow of (Ev).

Lemma (spectral analysis of $\widetilde{\mathcal{E}}''(0,0)$)

The spectrum of $\widetilde{\mathcal{E}}''(0,0)$ *is discrete,* $\sigma = \{0, \mu_1, \mu_2, \ldots\}$, *where* $0 < \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \mu_4 \leqq \cdots$, *and*

$$\ker \widetilde{\mathcal{E}}''(0,0) = \operatorname{span}\left\{ \begin{pmatrix} f_z \\ 0 \end{pmatrix} \right\}.$$

Moreover, there is $\beta > 0$ *such that for every* $\vec{h}$ *satisfying* $\vec{h} \perp (f_z, 0)^\top$ *we obtain*

$$\langle \widetilde{\mathcal{E}}''(0,0)\vec{h}, \vec{h} \rangle \geqq \beta \|\vec{h}\|_Z^2$$

Theorem (stability of zero solution)

The trivial solution $\vec{w} \equiv \vec{0}$ *for (Ev) is stable in* $\mathcal{Z}$ *by the periodic flow generated by the evolution equation (Ev), that is, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for* $\vec{w}_0 \in \mathcal{Z}$*, and* $\|\vec{w}_0\|_{\mathcal{Z}} < \delta$ *we have that the solution* $\mathbf{w}(t)$ *with* $\mathbf{w}(0) = \vec{w}_0$ *satisfies* $\mathbf{w}(t) \in Z$ *and*

$$\|\mathbf{w}(t)\|_{\mathcal{Z}} < \varepsilon.$$

Theorem (orbital stability)

The rotational subluminal traveling wave profile is orbitally stable in Z *by the flow generated by the two-dimensional sine-Gordon: For every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for* $u_0 = u_0(\cdot,\cdot) \in \mathcal{X}_\pm(T) \times H^1_{\mathrm{per}}([0,L])$ *with*

$$\int_0^T u_0(z,y)dz = 0, \quad \textit{for all } y \in [0,L],$$

and $u_1 \in \mathcal{N}$ *satisfying*

$$\|u_0 - F\|_{H^1_{\mathrm{per}}([0,T]\times[0,L])} + \|c\partial_z u_0 + u_1\|_{L^2_{\mathrm{per}}([0,T]\times[0,L])} < \delta,$$

then the solution $u = u(z,y,t)$ *with initial conditions* $u(\cdot,\cdot,0) = u_0(\cdot,\cdot)$ *and* $u_t(\cdot,\cdot,0) = u_1(\cdot,\cdot)$ *satisfies for all* t*:*

Theorem ((continued))

$$\begin{cases} t \to u(\cdot + ct, \cdot, t) - F(\cdot, \cdot) \in H^1_{\text{per}}([0,T] \times [0,L]) \\ t \to c\partial_z u(\cdot + ct, y, t) + u_t(\cdot + ct, y, t) \in L^2_{\text{per}}([0,T] \times [0,L]), \end{cases}$$

Furthermore,

$$\|u(\cdot+\gamma,\cdot,t)-F(\cdot,\cdot)\|_{H^1_{\text{per}}([0,T]\times[0,L])}+\|c\partial_z u(\cdot,\cdot,t)+u_t(\cdot,\cdot,t)\|_{L^2_{\text{per}}([0,T]\times[0,L])}<\varepsilon,$$

for all $t > 0$.

Remark: Here the modulation parameter γ is given explicitly by $\gamma(t) = ct$. Moreover, we have

$$t \in \mathbb{R} \to u(\cdot, y, t) \in \mathcal{X}_{\pm}(T), \quad \text{and} \quad \int_0^T u(z + ct, y, t)\,dz = 0$$

for all y fixed and $t \in \mathbb{R}$.

Muito obrigado!