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ON THE STABILITY OF DEGENERATE VISCOUS SHOCK PROFILES

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Abstract. In [16], we showed how to obtain L^p-decay rates for zero-mass perturbations of degenerate scalar viscous shock waves using energy methods. The proof is based upon previous work by Matsumura and Nishihara [12], by extending their weighted energy estimates to L^p-norms, and by obtaining sharp decay rates with the aid of basic interpolation inequalities. This contribution summarizes the results in [16] and illustrates the use of interpolation inequalities to obtain decay rates by analyzing the L^2 case as an example.

1. **Introduction.** Consider a scalar viscous conservation law,

$$u_t + f(u)_x = (b(u)u_x)_x, \tag{1}$$

where $f, b \in C^2$, $b > 0$, and $(x,t) \in \mathbb{R} \times [\,0, +\infty)$. This paper pertains to the stability of viscous shock wave solutions to equation (1) of form $u(x,t) = \bar{u}(x - st)$, where $\bar{u}$ satisfies

$$(b(\bar{u})\bar{u}')' = f(\bar{u})' - s\bar{u}',$$
$$\bar{u}(\xi) \to u_\pm, \;\text{ as }\; \xi \to \pm\infty.$$

Here, $' = d/d\xi$ denotes differentiation with respect to $\xi := x - st$, and s is the shock speed. We assume that the triple $(u_+, u_-, s) \in \mathbb{R}^3$, with $u_- \neq u_+$ (say $u_+ < u_-$), is a generalized shock front [10] satisfying the classical Rankine-Hugoniot jump condition and the generalized entropy condition (cf. [11, 17]). We pay particular attention to the case where the flux function $u \mapsto f(u)$ changes its convexity in $u \in (u_+, u_-)$. Such a hypothesis allows us to consider *sonic (or degenerate) shocks*, namely, waves whose speed matches one of the characteristic speeds with $s = f'(u_+)$ or $s = f'(u_-)$.

The stability of degenerate viscous shocks has been addressed by many authors. One of the first approaches consisted on obtaining *a priori* energy estimates for the perturbations (cf. [13, 14, 15, 12]). This approach is usually called the energy method. In particular, the seminal paper of Matsumura and Nishihara [12] introduced a suitable, asymmetric weight function, not bounded on the sonic side, which accomodates properly on the compressive side, yielding the right sign in the weighted energy estimates. Another approach, developed a few years later by

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Howard [8, 7] (and which generalizes the method introduced by Howard [6], and by Howard and Zumbrun [18] for classical shocks), follows pointwise bounds on the Green's function for the linearized operator around the wave. These techniques offered more information on the asymptotic behavior of the solutions than energy methods, yielding sharper decay rates in all L^p spaces (see [8] for details). The application of the pointwise Green's function method to degenerate shocks is, however, more complicated than in the classical case, due to the fact that degenerate profiles decay algebraically, in contrast to the exponential decay of Laxian shocks. This makes the Evans function non-analytic near $\lambda = 0$ (cf. [8, 7, 18]). Thus, the standard estimates must be replaced by difficult and very technical ODE estimates.

One arguable drawback of the energy approach has been its inability to provide sharp estimates on the rate at which perturbations decay. In [16], however, we show that a sharp rate of decay can be obtained by energy methods very similar to those applied by Mei [13, 14], Matsumura-Nishihara [12], and Nishikawa [15]. Here we summarize the results and illustrate the use of interpolation inequalities to get sharp rates of decay by examining the L^2 case (see §3). For further details the reader is referred to [16].

2. **Preliminaries and main result.**

By translation invariance, we normalize the flux function f such that the profile is stationary (i.e. $s = 0$), obeying the equation

$$b(\bar{u})\bar{u}_x = f(\bar{u}). \tag{2}$$

Therefore we summarize our assumptions as follows:

$$
\begin{aligned}
f, b \in C^2,\ b > 0, & \qquad \text{(regularity and positive diffusion)}, & \text{(A1)}\\
f(u_-) = f(u_+) = 0, & \qquad \text{(Rankine-Hugoniot condition)}, & \text{(A2)}\\
f(u) < 0 \ \text{ for all } \ u \in (u_+, u_-), & \qquad \text{(generalized entropy condition)}. & \text{(A3)}
\end{aligned}
$$

The generalized entropy condition (A3) implies the non-strict condition $f'(u_+) \leq 0 \leq f'(u_-)$, which allows sonic waves with $f'(u_+) = 0$ or $f'(u_-) = 0$. For concreteness, we assume that the shock is sonic on the positive side, namely, that $f'(u_+) = 0$. Whence, we rewrite assumption (A3) as

$$f(u) < 0, \ \text{ for all } \ u \in (u_+, u_-), \quad \text{and,} \quad f'(u_+) = 0 < f'(u_-). \tag{A3'}$$

2.1. **Existence and structure of profiles.**

The following existence result is well-known (cf. [17, 12, 2]; for a proof, see Howard [8], Section 2).

Proposition 2.1. *Under hypotheses (A1) - (A3'), let us define $\theta := \min\{k \in \mathbb{Z}^+ : (d^k f/du^k)(u_+) \neq 0\} \geq 1$, as the degree of degeneracy of the shock. Then there exists a traveling wave solution $\bar{u}$ of (2) with $\bar{u}(\pm\infty) = u_\pm$, unique up to translations. Moreover, $\bar{u}$ is monotone decreasing, $\bar{u}_x < 0$, and $\bar{u}$ and its derivatives decay like*

$$
\begin{aligned}
|\partial_x^j (\bar{u}(x) - u_-)| &\leq C e^{-c|x|}, \qquad \text{as } \ x \to -\infty,\\
|\partial_x^j (\bar{u}(x) - u_+)| &\leq C |x|^{-1/\theta}, \qquad \text{as } \ x \to +\infty,
\end{aligned}
$$

for $j = 0, 1, 2$, and some uniform $C > 0$.

2.2. **Perturbation equations.** We restrict our analysis to the class of perturbations with zero-mass, and choose δ such that $\int_{\mathbb{R}}(u_0(x) - \bar{u}(x-\delta))\,dx = 0$. This allows us to write the perturbation as $u(x,t) - \bar{u}(x-\delta) = v_x(x,t)$, for some function $v(\cdot,t)$ in L^2, i.e., we can integrate the equation [3, 5]. Suppose, without loss of generality, that $\delta = 0$; this yields

$$\int_{\mathbb{R}} (u_0(x) - \bar{u}(x))\,dx = 0. \tag{3}$$

In view of last observations, substitute now $u(x,t) = v_x(x,t) + \bar{u}(x)$ into (1), integrate in $(-\infty, x)$ and use the profile equation (2) to obtain the perturbation equation $v_t = b(\bar{u})v_{xx} - a(x)v_x + F$, where $a(x) = f'(\bar{u}) - b(\bar{u})_x$, and

$$F = -(f(v_x+\bar{u}) - f(\bar{u}) - f'(\bar{u})v_x) + (b(v_x+\bar{u}) - b(\bar{u}) - b'(\bar{u})v_x)(\bar{u}_x + v_{xx}) + b'(\bar{u})v_x v_{xx},$$

comprises the nonlinear terms. Therefore, after these reformulations, the Cauchy problem for the perturbation v is written as follows,

$$v_t = b(\bar{u})v_{xx} - a(x)v_x + F, \qquad \text{for } \ (x,t) \in \mathbb{R}\times(0,+\infty), \tag{4}$$

$$v(x,0) = v_0(x) = \int_{-\infty}^{x} (u_0(y) - \bar{u}(y))\,dy, \qquad \text{for } \ x \in \mathbb{R}. \tag{5}$$

2.3. **The Matsumura-Nishihara weight function.** Matsumura and Nishihara [12] introduced the following weight function,

$$\eta(x) = \bar{\eta}(\bar{u}(x)), \qquad \bar{\eta}(u) = \frac{(u-u_+)(u-u_-)}{f(u)} > 0, \qquad u \in (u_+, u_-). \tag{6}$$

which behaves like

$$\eta \sim \langle x \rangle_+ = \begin{cases} (1+x^2)^{1/2}, & x \geq 0, \\ 1, & x < 0. \end{cases}$$

(See [12] for details.) In particular we have that η is bounded below, $\eta \geq \bar{C}^{-1} > 0$, for all x, and with some uniform $\bar{C} > 0$. Note, however, that it is not bounded above as it blows up on the sonic side when $x \to +\infty$. The remarkable property of the weight function (6) is that it leads to the right sign of the term

$$\Phi(x) = ((b(\bar{u})\eta)_x + a(x)\eta)_x = ((d/du)(f(u)\bar{\eta}(u))_{|u=\bar{u}})_x = -2|\bar{u}_x| < 0,$$

for all $x \in \mathbb{R}$ (see [12]). It can also be shown that $|\eta_x| \leq C\eta$ for some uniform $C > 0$.

Finally, let us specify some notation. $W^{m,p}$ denotes the standard Sobolev spaces in $\mathbb{R}$. In terms of the weight function, for each $1 \leq p < +\infty$, L^p_η denotes the space of measurable functions u such that $\eta^{1/p}u \in L^p$, or that $\|u\|^p_{L^p_\eta} := \int_{\mathbb{R}} \eta |u|^p\,dx < +\infty$.

2.4. **Main result.** We are ready to state the main theorem in [16].

Theorem 2.2 ([16])**.** *Under assumptions (A1) - (A3'), with $u_+ < u_-$, let $\bar{u}$ be the traveling wave solution to (1) of Proposition 2.1. Suppose that the zero-mass condition (3) holds, and that*

$$v_0 := \int_{-\infty}^{x} (u_0(x) - \bar{u}(x))\,dx \, \in \, Z_{\eta,p},$$

where $Z_{\eta,p} = L^1_\eta \cap L^2_\eta \cap L^p_\eta \cap W^{2,p}$, for some $2 \leq p < +\infty$, and η denotes the Matsumura-Nishihara weight function (6). Then there exists a positive constant $\hat{\epsilon} > 0$ such that if $\|v_0\|_{Z_{\eta,p}} < \hat{\epsilon}$, then the Cauchy problem for equation (1) with *initial condition* $u(0) = u_0$ *has a unique global solution* $u - \bar{u} \in C([0, +\infty]; W^{1,p})$ *satisfying*

$$\|u - \bar{u}\|_{L^p} \leq CME_0 t^{-1/2}(1+t)^{-\frac{1}{2}(1-1/p)}, \tag{7}$$

$$\|u - \bar{u}\|_{L^\infty} \leq CME_0 t^{-1/2-1/2p}(1+t)^{-\frac{1}{2}(1-1/p)}, \tag{8}$$

for all $0 < t < +\infty$*, where* $E_0 = \|v_0\|_{L^1_\eta} + \|v_0\|_{L^p_\eta} + \|v_0\|^2_{L^2_\eta}$*, and with uniform constant* $M > 0$*.*

Like that of Matsumura and Nishihara [12], this result applies to zero-mass perturbations only and require very rapidly decaying data, as v_0 must belong to the weighted space $Z_{\eta,p}$. Observe, however, that the decay rates (7) are sharp. It is also to be noted that, unlike the Green's function method, the analysis works for all degrees of degeneracy.

3. **Decay rates.** In this section we show how to use interpolation inequalities to obtain sharp rates of decay. The following inequality is the weighted norm version of the interpolation inequality by Escobedo and Zuazua [1] (Lemma 1, pg. 129). The proof in [16] follows [1] closely, with the appropriate adaptations to the weighted spaces under consideration. Notably, the original inequality remains valid in weighted spaces, even though the function η is not bounded above on the sonic side.

Lemma 3.1 (Weighted interpolation inequality [1, 16])**.** *For each* $2 \leq p < +\infty$ *there exists some constant* $C = C(p) > 0$ *such that*

$$\|u\|^{p(p+1)/(p-1)}_{L^p_\eta} \leq C\|u\|^{2p/(p-1)}_{L^1_\eta} \|(|u|^{p/2})_x\|^2_{L^2_\eta}, \tag{9}$$

for every $u \in W^{2,p} \cap L^p_\eta \cap L^1_\eta$*, with* $u_x \in W^{2,p} \cap L^p_\eta$*, where* η *denotes the Matsumura-Nishihara weight function.*

We illustrate how to obtain this inequality in the L^2 (non-weighted) case (for the general proof, see [16]). By the Sobolev inequality $\|u\|^2_\infty \leq 2\|u\|_{L^2}\|u_x\|_{L^2}$, we readily get

$$\|u\|^8_{L^2} = \Big(\int_{\mathbb{R}} u^2 \, dx \Big)^4 \leq \|u\|^4_\infty \|u\|^4_{L^1} \leq 4\|u\|^2_{L^2}\|u_x\|^2_{L^2}\|u\|^4_{L^1}.$$

This immediately yields $\|u\|^6_{L^2} \leq C\|u_x\|^2_{L^2}\|u\|^4_{L^1}$ which is the inequality (9) for $p = 2$.

For simplicity, let us explain to obtain sharp decay rates in L^2. We begin with an observation. Let $\rho(t) \geq 0$ be of class C^1 in $t > 0$, such that $d\rho/dt \leq C\rho^\beta$ for some $C > 0$, $\beta > 1$, and with $\rho(0) = \rho_0 > 0$. Then clearly $\rho(t) \leq \xi(t)$, a.e. in $t > 0$, where $\xi = \xi(t)$ is the solution to $d\xi/dt = -C\xi^\beta$, and $\xi(0) = \rho_0$. Now suppose that for $0 < t < T \leq +\infty$, and some $C_0, C_1 > 0$, there hold

$$\frac{d}{dt}\|u(t)\|^2_{L^2} \leq -C_1\|u_x\|^2_{L^2}, \quad \text{and} \ \ \|u(t)\|_{L^1} \leq C_0.$$

Use the interpolation inequality to get

$$\frac{d}{dt}\|u\|^2_{L^2} \leq -C_1\|u_x\|^2_{L^2} \leq -\frac{C_1}{C}\|u\|^6_{L^2}\|u\|^{-4}_{L^1} \leq -C_2\|u\|^6_{L^2},$$

where $C_2 = C_1/(CC_0^4)$. It is easy to verify that $\xi = \big(2C_2 t + \|u(0)\|^{-4}_{L^2}\big)^{-1/2}$ is the solution to $d\xi/dt = -C_2\xi^3$, $\xi(0) = \|u(0)\|^2_{L^2}$ (here $\beta = 3$). Hence, we clearly have

$\xi(t)^{-2} \geq C_3(1+t)$, with $C_3 = \min\{2C_2, \|u(0)\|_{L^2}^{-4}\}$, and by the observation above we obtain

$$\|u(t)\|_{L^2}^2 \leq \xi(t)^2 \leq C_3^{-1/2}(1+t)^{-1/2} \leq \tilde{C}(C_0 + \|u(0)\|_{L^2})^2(1+t)^{-1/2}.$$

Therefore we have proved the following decay rate:

$$\|u(t)\|_{L^2} \leq C(1+t)^{-1/4},$$

where the constant is $C = O(C_0 + \|u(0)\|_{L^2})$. In a similar fashion, and using the weighted interpolation inequality (9), it is possible to prove the following result (see [16] for details):

Lemma 3.2. *Let $2 \leq p < +\infty$, and suppose that u is the solution to a certain evolution (linear or nonlinear) equation, which satisfies the bounds*

$$\frac{d}{dt}\|u(t)\|_{L^p_\eta}^p \leq -C_1\|(|u|^{p/2})_x(t)\|_{L^2_\eta}^2, \tag{10}$$

$$\|u(t)\|_{L^1_\eta} \leq C_0, \tag{11}$$

for all $0 < t < T \leq +\infty$, and uniform constants $C_1, C_0 > 0$. Then, there exists a positive constant $\bar{C} > 0$ such that

$$\|u(t)\|_{L^p_\eta} \leq \bar{C}(1+t)^{-\frac{1}{2}(1-1/p)},$$

for all $0 < t < T$. Moreover, the constant $\bar{C}$ is of order $\bar{C} = \mathcal{O}(C_0 + \|u(0)\|_{L^p_\eta})$.

4. **Energy estimates.** According to custom, the global existence and the stability are proved by a continuation argument based on a local existence result combined with the corresponding *a priori* energy estimates. Assume $2 \leq p < +\infty$ is fixed. Let us define the suitable space for solutions as $Z_{\eta,p} = W^{2,p} \cap L^p_\eta \cap L^1_\eta \cap L^2_\eta$, and $X_{\eta,p}(0,T) = \{v \in C([0,T]; Z_{\eta,p}),\ v_x \in L^2([0,T]; Z_{\eta,p})\}$ with $0 < T \leq +\infty$. Using the variation of constants formula and by a standard contraction mapping argument it is possible to prove the following short-time existence result.

Proposition 4.1 (Local existence)**.** *For any $\epsilon_0 > 0$ there exists a positive constant T_0 depending on ϵ_0 such that if $v_0 \in Z_{\eta,p}$ and $\|v_0\|_{Z_{\eta,p}} \leq \epsilon_0$, then the Cauchy problem (4) and (5) has a unique solution $v \in X_{\eta,p}(0,T_0)$ satisfying $\|v(t)\|_{Z_{\eta,p}} < 2\epsilon_0$ for each $0 \leq t \leq T_0$.*

4.1. **The basic energy estimate.** Here we describe the energy estimates for the perturbations obtained in [16]. The proofs follow standard energy estimates in L^p spaces.

Lemma 4.2 (Basic energy estimate [16])**.** *Let $2 \leq p < +\infty$, and let $v(t) \in X_{\eta,p}(0,T)$ be a solution to (4) for some $T > 0$. Then, for each $0 \leq t \leq T$ there holds*

$$\frac{1}{p}\|v(t)\|_{L^p_\eta}^p + \frac{4(p-1)}{p^2}\int_0^t \|b(\bar{u})^{1/2}(|v(\tau)|^{p/2})_x\|_{L^2_\eta}^2\, d\tau + \frac{2}{p}\int_0^t\int_{\mathbb{R}} |\bar{u}_x||v(\tau)|^p\, dx\, d\tau =$$
$$= \frac{1}{p}\|v(0)\|_{L^p_\eta}^p + \int_0^t\int_{\mathbb{R}} \eta F v(\tau)|v(\tau)|^{p-2}\, dx\, d\tau.$$

Let us now define $R(t) := \sup_{\tau\in[0,t]} \|v(\tau)\|_{Z_{\eta,p}}$ for each $t \in [0,T]$, with $T > 0$ fixed.

Lemma 4.3 ([16]). *There exists $\epsilon_1 > 0$ sufficiently small such that if $R(t) < \epsilon_1$ for $t \in [0, T]$ then we have the estimate*

$$\|v(t)\|_{L^p_\eta}^p + \hat{C}_1 \int_0^t \|(|v(\tau)|^{p/2})_x\|_{L^2_\eta}^2 \, d\tau \,\leq\, \|v(0)\|_{L^p_\eta}^p, \tag{12}$$

for some $\hat{C}_1 > 0$ depending on p and ϵ_1, and for all $0 \leq t \leq T$.

Two immediate corollaries follow.

Corollary 4.4. *Specializing (12) to the case $p = 2$ we have that if $R(t) < \epsilon_1$ for all $0 \leq t \leq T$ then*

$$\|v(t)\|_{L^2_\eta}^2 + \hat{C}_1 \int_0^t \|v_x(\tau)\|_{L^2_\eta}^2 \, d\tau \,\leq\, \|v(0)\|_{L^2_\eta}^2.$$

Corollary 4.5. *If $v \in X_{\eta,p}(0,T)$ is a solution with $R(t) \leq \epsilon_1$ for $0 \leq t \leq T$, then*

$$\frac{d}{dt}\|v(t)\|_{L^p_\eta}^p + C_1 \|(|v|^{p/2})_x(t)\|_{L^2_\eta}^2 \,\leq\, 0,$$

for some $C_1 > 0$.

4.2. **L^1_η-bound and decay rates.** The boundedness of $\|v(t)\|_{L^1_\eta}$, plays a key role in Lemma 3.2. This a remarkable property of the solutions to (4), and of the Matsumura-Nishihara weight function. (See [16] for details.)

Lemma 4.6. *Assuming $R(t) < \epsilon_1$, with $\epsilon_1 > 0$ just as in Lemma 4.3, the following estimate holds*

$$\|v(t)\|_{L^1_\eta} \leq C(\|v(0)\|_{L^1_\eta} + \|v(0)\|_{L^2_\eta}^2),$$

for some $C > 0$, all $0 \leq t \leq T$.

The previous observations and the decay rates of the previous section readily imply the following

Corollary 4.7. *Let $v \in X_{\eta,p}(0,T)$ be a solution to (4), for some $T > 0$. If $R(t) < \epsilon_1$ for all $0 \leq t \leq T$, then v satisfies the decay rate*

$$\|v(t)\|_{L^p_\eta}^p \leq C E_0^p (1+t)^{-(p-1)/2}, \tag{13}$$

with $E_0 := \|v(0)\|_{L^1_\eta} + \|v(0)\|_{L^p_\eta} + \|v(0)\|_{L^2_\eta}^2$.

Proof. Since $R(t) < \epsilon_1$, we may apply Lemma 4.6 and Corollary 4.5 to conclude that properties (10) and (11) hold with $C_0 = \mathcal{O}(\|v(0)\|_{L^1_\eta} + \|v(0)\|_{L^2_\eta}^2)$. Then, by Lemma 3.2 we obtain the desired decay rate (13). □

4.3. **Higher order estimates.** The L^p estimates for the derivatives cannot be controlled as in the L^2 case (where $\|(|u|^{p/2})_x\|$ is equivalent to $\|u_x\|$ and there is a natural way to construct a decreasing norm). Thus, we follow the general method of [9] instead. (For details, the reader is referred to [16].)

Lemma 4.8 (Higher order estimates [16]). *Suppose $v \in X_{\eta,p}(0,T)$, $2 \leq p < +\infty$, with $0 < T \leq 1$ solves (4). Then there exists $\epsilon_2 > 0$, sufficiently small, such that if $R(t) < \epsilon_2$ for $0 \leq t \leq T \leq 1$, then there hold the estimates*

$$\begin{aligned}\frac{1}{p} t^\alpha (1+t)^\beta \|v_x(t)\|_{L^p}^p &+ \hat{C}_2 \int_0^t \tau^\alpha (1+\tau)^\beta \|(|v_x|^{p/2})_x(\tau)\|_{L^2_\eta}^2 \, d\tau \\ &\leq C E_0^p t^{\alpha - p/2} (1+t)^{\beta - \frac{1}{2}(p-1)},\end{aligned}$$

$$\frac{1}{p}t^\gamma(1+t)^\delta\|v_{xx}(t)\|^p_{L^p} + \hat{C}_3\int_0^t \tau^\gamma(1+\tau)^\delta\|(|v_{xx}|^{p/2})_x(\tau)\|^2_{L^2_\eta}\,d\tau$$
$$\leq CE_0^p t^{\gamma-p}(1+t)^{\delta-\frac{1}{2}(p-1)},$$

for $0 \leq t \leq T$, where $\alpha, \beta > 0$ satisfy $\alpha > p/2$, $\beta > \frac{1}{2}(p-1)$; $\gamma, \delta > 0$ satisfy $\gamma > p$, $\delta > \frac{1}{2}(p-1)$, and $C, \hat{C}_2, \hat{C}_3 > 0$ are constants depending on $p, \epsilon_2, \bar{u}, \alpha$, β, γ and δ. Moreover, for $0 < t \leq T$ there holds the decay rates

$$\|v_x(t)\|^p_{L^p} \leq CE_0^p t^{-p/2}(1+t)^{-\frac{1}{2}(p-1)}. \tag{14}$$

$$\|v_{xx}(t)\|^p_{L^p} \leq CE_0^p t^{-p}(1+t)^{-\frac{1}{2}(p-1)}. \tag{15}$$

The additional assumption $T \leq 1$ means no loss of generality, as the local existence time in Proposition 4.1 can be chosen as $\hat{T}_0 = \min\{1, T_0(\epsilon_0)\}$.

5. **Stability and proof of Theorem 2.2.** We apply the previous *a priori* estimates to show stability and to prove the main Theorem. This is achieved by a continuation argument to obtain global existence of solutions to the Cauchy problem, paying special attention to the fact that estimates for the derivatives (14) and (15) apply only within time intervals of measure one. Thus, the proof slightly deviates from the standard argument. Notably, the energy E_0 does not involve norms of the derivatives, and this allows to extend the decay rates globally in time.

Theorem 5.1 ([16]). *Suppose $v_0 \in Z_{\eta,p}$, with $2 \leq p < +\infty$. Then there exists a positive constant $\hat{\epsilon} > 0$ such that if $\|v_0\|_{Z_{\eta,p}} < \hat{\epsilon}$, then the Cauchy problem (4) and (5) has a unique global solution $v \in X_{\eta,p}(0,+\infty)$ which satisfies the following estimates*

$$\|v(t)\|_{L^1} \leq ME_0,$$
$$\|v(t)\|_{L^p} \leq ME_0(1+t)^{-\frac{1}{2}(1-1/p)},$$
$$\|v(t)\|_{L^2} \leq ME_0(1+t)^{-\frac{1}{4}},$$
$$\|v_x(t)\|_{L^p} \leq ME_0 t^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}(1-1/p)},$$
$$\|v_{xx}(t)\|_{L^p} \leq ME_0 t^{-1}(1+t)^{-\frac{1}{2}(1-1/p)},$$

for all $0 < t < +\infty$, with some uniform $M > 0$, and where $E_0 = \|v_0\|_{L^1_\eta} + \|v_0\|_{L^p_\eta} + \|v_0\|^2_{L^2_\eta}$.

The main Theorem 2.2 is now a direct consequence of Theorem 5.1.

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