Orbital stability of standing waves for the nonlinear Schrödinger equation with attractive delta potential and double power repulsive nonlinearity

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Introduction

Nonlinear Schrödinger equation (NLS)

NLS equation with attractive delta potential and repulsive double power nonlinearity:

$$iu_t + u_{xx} + Z\delta(x)u + \lambda_1 u|u|^{p-1} + \lambda_2 u|u|^{2p-2} = 0$$

- Unknown: $u = u(x,t) \in \mathbb{C}$, for $x, t \in \mathbb{R}$.
- Parameters: $\lambda_1 \leq 0$, $\lambda_2 < 0$, $Z > 0$, $p > 1$.
- $\delta : H^1(\mathbb{R}) \to \mathbb{C}$, $\langle \delta, g \rangle = g(0)$ (Dirac delta centered at $x = 0$.)
- Linear interaction: $\partial_x^2 + Z\delta(x)$.
- Nonlinear term: $\lambda_1 u|u|^{p-1} + \lambda_2 u|u|^{2p-2}$.
- $i^2 = -1$.

Physical applications

- We recall that the general NLS model

$$iu_t + u_{xx} + V(x)u + f(|u|^2)u = 0,$$

  represents a trapping (wave-guiding) structure for light beams, induced by an inhomogeneity of the local refractive index.
- The delta-function term $V(x) = Z\delta(x)$ represents a narrow trap which is able to capture broad solitonic beams.
- It models a spatially localized point defect of the medium in which the soliton travels (localized attractive "impurity").
- The non linear term $f(x) = \lambda_1 x^{(p-1)/2} + \lambda_2 x^{p-1}$ is well known in optical media.

Standing waves

Standing waves are solutions to the NLS model of the form

$$u(x,t) = e^{-i\omega t}\phi(x),$$

where $\omega \in \mathbb{R}$ and the profile of the wave $\phi : \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} \phi'' + Z\delta(x)\phi + \omega\phi + f(|\phi|^2)\phi = 0, \\ \phi \in H^1(\mathbb{R}), \end{cases} \tag{ODE}$$

where $f = f(\cdot)$ is an arbitrary function satisfying

$$\begin{aligned} f \in C^1((0,+\infty);\mathbb{R}) \quad \text{with } f(0) = 0, \\ f'(x) < 0 \quad \text{for all } x > 0. \end{aligned} \tag{H$_f$}$$

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Example: if $1 < p < \infty$, $\lambda_1 \leq 0$ and $\lambda_2 < 0$ then

$$f(x) = \lambda_1 x^{(p-1)/2} + \lambda_2 x^{p-1}$$

satisfies (H_f).

δ-interaction quantum operator

The δ-interaction quantum operator A_Z is defined as

$$A_Z := -\partial_x^2 - Z\delta(x)$$

$$\begin{cases} A_Z f(x) = -f''(x), \qquad x \neq 0, \\ D(A_Z) = \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \backslash \{0\}) : f'(0+) - f'(0-) = -Zf(0)\}, \end{cases}$$

Orbital stability

Definition (orbital stability)

The standing wave $e^{-i\omega t}\phi_\omega$ is orbitally stable by the flow of the NLS equation n $H^1(\mathbb{R})$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u_0 - \phi_\omega\|_{H^1} < \delta$ then

$$\inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\phi_\omega\|_{H^1} < \varepsilon, \qquad \text{for all } t \in \mathbb{R},$$

where $u(t)$ denotes the solution to the NLS equation with initial data $u(0) = u_0 \in H^1(\mathbb{R})$. Otherwise, $e^{-i\omega t}\phi_\omega$ is said to be orbitally unstable in $H^1(\mathbb{R})$.

Main theorem

Theorem (Angulo Pava, Hernandez Melo, P (2019))

Let $1 < p < \infty$, $\lambda_1 \leq 0$, $\lambda_2 < 0$ and $Z > 0$ in the NLS equation. Then for all values of $\omega < 0$ satisfying

$$-\frac{p\lambda_1^2}{(p+1)^2\lambda_2} < -\omega < \frac{Z^2}{4}$$

the family of standing wave solutions, $u(x,t) = e^{-i\omega t}\phi_\omega$, with ϕ_ω given by

$$\phi_\omega = \left[\frac{\alpha}{-\omega} + \frac{\sqrt{v}}{-\omega}\sinh\left((p-1)\sqrt{-\omega}\left(|x| + R_1^{-1}\left(\frac{Z}{2\sqrt{-\omega}}\right)\right)\right)\right]^{-\frac{1}{p-1}}$$

where $v = \omega\beta - \alpha^2$, are orbitally stable solutions in $H^1(\mathbb{R})$ under the flow of the NLS equation.

Previous results

- Case $Z = 0$: Ohta (1995), double power nonlinearity; Maeda (2008), multiple power nonlinearity.
- Repulsive δ potential: Fukuizumi, Jeanjean (2008)
- Attractive δ potential: Fukuizumi et al. (2008)
- Kaminaga, Ohta (2009): attractive δ with repulsive single power nonlinearity.

Multibody interactions of same sign (repulsive, double power nonlinearity) appear in the study of Bose-Einstein condensates: Brazhnyi, Konotop (2004); Belobo Belobo et al. (2014); Kamchatnov, Salerno (2009); Kamchatnov, Korneev (2010) (dark solitons).

Concentration-compactness method: Cazenave, Lions (1982)

- **The Cauchy problem**: The initial value problem associated to the NLS equation is globally well-posed in $H^1(\mathbb{R})$ for $1 < p < +\infty$, $\lambda_1 \leq 0$, $\lambda_2 < 0$ and $Z > 0$.
- **Existence of profile solution** ϕ_ω: for parameter values $p, \lambda_1, \lambda_2, Z$ and ω satisfying the assumptions there exists a profile solution ϕ_ω of the elliptic equation (ODE) (explicit construction).
- **The stationary problem**: The set $\mathscr{A}_\omega$ of non-trivial solutions of the equation for the profiles in $H^1(\mathbb{R})$ will be characterized, via uniqueness, by

$$\mathscr{A}_\omega = \{v : G'_\omega(v) = 0, v \neq 0\} = \{e^{i\theta}\phi_\omega : \theta \in \mathbb{R}\},$$

where

$$G_\omega(v) = E(v) - \frac{\omega}{2}\|v\|_{L^2}^2,$$

$$E(v) := \frac{1}{2}\|v_x\|_{L^2}^2 - \frac{Z}{2}|v(0)|^2 - \frac{\lambda_1}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{\lambda_2}{2p}\|v\|_{L^{2p}}^{2p},$$

for $v \in H^1(\mathbb{R})$.

Concentration-compactness method (ii)

- **The minimization problem**: For $p, \lambda_1, \lambda_2, Z$ and ω satisfying the assumptions of our main theorem, the quantity

$$m(\omega) = \inf\{ G_\omega(v) : v \in H^1(\mathbb{R})\},$$

satisfies the following properties:

(a) (boundedness below) $-\infty < m(\omega) < 0$; and,

(b) (compactness) any sequence $h_n \in H^1(\mathbb{R})$ such that $\lim_{n\to\infty} G_\omega(h_n) = m(\omega)$ admits a subsequence converging to some $h \in H^1(\mathbb{R})$ with $G_\omega(h) = m(\omega)$.

Local and global well posedness for the NLS

Preliminaries (i)

The formal expression $A_Z := -\partial_x^2 - Z\delta(x)$ represents all the self-adjoint extensions (von Neumann theory) associated to the following closed, symmetric, densely defined linear operator:

$$\begin{cases} A_0 = -\partial_x^2 \\ D(A_0) = \{g \in H^2(\mathbb{R}) : g(0) = 0\}. \end{cases}$$

More precisely, the quantum operator $A_Z = -\partial_x^2 - Z\delta(x)$ is given by

$$\begin{cases} A_Z f(x) = -f''(x) \qquad x \neq 0, \\ D(A_Z) = \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{0\}) : f'(0+) - f'(0-) = -Zf(0)\}. \end{cases}$$

Preliminaries (ii)

Upon application of the First Representation Form Theorem (cf. Kato), it is possible to show that the associated form to A_Z is given by

$$F_Z[u,v] = \mathrm{Re}\int_{-\infty}^{+\infty} u'(x)\overline{v'(x)}dx - Z\,\mathrm{Re}(u(0)\overline{v(0)}),$$

where $(u,v) \in D(F_Z) = H^1(\mathbb{R}) \times H^1(\mathbb{R})$. The bilinear form defined above is closed and bounded below. In addition, operator $A_Z = -\partial_x^2 - Z\delta(x)$ can be extended as a linear bounded operator $u \to A_Z u$ from $H^1(\mathbb{R})$ to $H^{-1}(\mathbb{R})$. This action is defined by

$$\langle A_Z u, v\rangle = F_Z[u,v], \qquad \text{for } u,v \in H^1(\mathbb{R}).$$

Spectral properties of A_Z

- Essential spectrum: $\Sigma_{\text{ess}}(A_Z) = [0, +\infty)$, for all $Z \in \mathbb{R}$.
- Discrete spectrum:

$$\Sigma_{\text{dis}}(A_Z) = \begin{cases} \varnothing, & Z \leq 0, \\ \left\{-Z^2/4\right\}, & Z > 0, \end{cases}$$

For $Z > 0$, $\Psi_Z(x) = \sqrt{\frac{Z}{2}} e^{-\frac{Z}{2}|x|}$ is the normalized eigenfunction associated to the unique negative simple eigenvalue $-Z^2/4$. In addition, the operators A_Z are bounded from below:

$$\begin{cases} A_Z \geq -Z^2/4, & Z > 0, \\ A_Z \geq 0, & Z < 0 \end{cases}$$

Ref.: Albeverio, Gesztesy, Høegh-Kron (2005).

The Cauchy problem

Consider the Cauchy problem,

$$\begin{cases} iu_t - A_Z u + (\lambda_1 |u|^{p-1} + \lambda_2 |u|^{2p-2})u = 0, \\ u(0) = u_0 \in H^1(\mathbb{R}). \end{cases}$$

Ref. Cazenave, Courant LN, vol. 10 (2003).

Local well-posedness

Theorem (local well-posedness)

For any $u_0 \in H^1(\mathbb{R})$ *and* $Z \in \mathbb{R}$, *there exists* $T > 0$ *and a unique solution* $u \in C([-T,T]; H^1(\mathbb{R})) \cap C^1([-T,T]; H^{-1}(\mathbb{R}))$ *to the NLS equation with* $u(0) = u_0$ *such that*

$$\lim_{t \to T^-} \|u(t)\|_{H^1} = +\infty, \qquad \text{if } T < \infty.$$

Moreover, the solution $u(t)$ *satisfies conservation of charge and energy:*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \qquad E(u(t)) = E(u_0),$$

for all $t \in [-T,T]$, *where the energy functional* E *is defined as*

$$E(v) := \frac{1}{2}\|v_x\|_{L^2}^2 - \frac{Z}{2}|v(0)|^2 - \frac{\lambda_1}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{\lambda_2}{2p}\|v\|_{L^{2p}}^{2p},$$

for $v \in H^1(\mathbb{R})$.

Proof sketch

- The nonnegative self-adjoint operator $\mathscr{A} \equiv A_Z + \beta$ on the space $X = L^2(\mathbb{R})$, with $\beta = Z^2/4$ for $Z > 0$ and $\beta = 0$ for $Z \leq 0$, and domain $D(\mathscr{A}) = D(A_Z)$, induces a norm

$$\|u\|_{X_{\mathscr{A}}}^2 = \|u_x\|_{L^2}^2 + (\beta+1)\|u\|_{L^2}^2 - Z|u(0)|^2,$$

  which is equivalent to the usual norm in $H^1(\mathbb{R})$.

- The self-adjoint operator A_Z generates a strongly continuous group of unitary operators $T(t)g = e^{-itA_Z}g$.

- Duhamel integral

$$u(t) = T(t)u_0 + \int_0^t T(t-s)(\lambda_1|u(s)|^{p-1}u(s) + \lambda_2|u(s)|^{2p-2}u(s))ds$$

  Direct application of Thm. 3.7.1 in Cazenave.

□

Remark: Gagliardo-Nirenberg interpolation inequality

For any $p > 1$, $H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \cap L^{p+1}(\mathbb{R}) \cap L^{2p}(\mathbb{R})$, inasmuch as the Gagliardo-Nirenberg interpolation inequality (cf. Leoni, 2017) yields

$$\|u\|_{L^{p+1}} \leq C_1 \|u\|_{L^2}^{\theta_1} \|u_x\|_{L^2}^{1-\theta_1},$$
$$\|u\|_{L^{2p}} \leq C_2 \|u\|_{L^2}^{\theta_2} \|u_x\|_{L^2}^{1-\theta_2},$$

with uniform constants $C_j > 0$ and $\theta_1 = (p+2)/(2p+2) \in (0,1)$, $\theta_2 = (p+1)/2p \in (0,1)$.

Conservation of charge/energy

- Conservation of charge:

$$\frac{d}{dt}||u(t)||_{L^2}^2 = 2\mathrm{Re}\int_{\mathbb{R}} u_t \overline{u} dx$$
$$= 2\mathrm{Re}\int_{\mathbb{R}} -i(A_Z u)\overline{u} + if(|u|^2)|u|^2 dx = 0.$$

- Conservation of Energy: NLS equation can be written in the Hamiltonian form $u_t = -iE'(u(t))$, then

$$\frac{d}{dt}E(u(t)) = \mathrm{Re}\int_{\mathbb{R}} E'(u(t))\overline{u}_t dx = \mathrm{Re}\int_{\mathbb{R}} i|E'(u(t))|^2 dx = 0.$$

Auxiliary bound

Let us define the following C^1 functional in $H^1(\mathbb{R})$,

$$R(v) := \frac{1}{2}\|v_x\|_{L^2}^2 - \frac{Z}{2}|v(0)|^2 - \frac{\lambda_2}{2p}\|v\|_{L^{2p}}^{2p} = E(v) + \frac{\lambda_1}{p+1}\|v\|_{L^{p+1}}^{p+1}.$$

Lemma (Auxiliary bound)

Let $1 < p < \infty$, $\lambda_1 \leq 0$, $\lambda_2 < 0$ and $Z > 0$. Then there exists a uniform constant $C = C(p,Z) > 0$ such that

$$\frac{Z}{2}|v(0)|^2 \leq R(v) + C, \qquad \textit{for all } v \in H^1(\mathbb{R}).$$

Proof sketch

By Sobolev and Young's inequalities, for any $Z > 0$ there exists $C_1 = C_1(Z) > 0$ such that for $v \in H^1(\mathbb{R})$

$$Z|v(0)|^2 \leq \frac{1}{2}\|v_x\|_{L^2}^2 + C_1\|v\|_{L^2(-1,1)}^2.$$

Apply Hölder's and Young's inequalities to estimate

$$\|v\|_{L^2(-1,1)}^2 \leq 2^{(p-1)/p}\left(\int_{-1}^{1}|v|^{2p}\,dx\right)^{1/p} \leq \delta\|v\|_{L^{2p}(-1,1)}^{2p} + 2C_\delta,$$

for any $\delta > 0$. Since $\lambda_2 < 0$, choose $\delta = -\lambda_2/(2pC_1) > 0$ to obtain

$$Z|v(0)|^2 \leq \frac{1}{2}\|v_x\|_{L^2}^2 - \frac{\lambda_2}{2p}\|v\|_{L^{2p}}^{2p} + 2C_1 C_\delta.$$

□

Global well-posedness

Theorem (global well-posedness)

For every $p > 1$, $Z > 0$, $\lambda_1 \leq 0$ and $\lambda_2 < 0$ the Cauchy problem is globally well-posed in $H^1(\mathbb{R})$.

Proof. Let $u \in C([-T,T]; H^1(\mathbb{R})) \cap C^1([-T,T]; H^{-1}(\mathbb{R}))$ be the local solution to the Cauchy problem for $t \in (-T,T)$.

$$\begin{aligned}\frac{1}{2}\|u_x\|_{L^2}^2 &= E(u) + \frac{Z}{2}|u(t)|^2 + \frac{\lambda_1}{p+1}\|u\|_{L^{p+1}}^{p+1} + \frac{\lambda_2}{2p}\|u\|_{L^{2p}}^{2p} \\ &\leq E(u(t)) + \frac{Z}{2}|u(t)|^2 \\ &\leq E(u(t)) + R(u(t)) + C\end{aligned}$$

Thus, we arrive at

$$\frac{1}{2}\|u_x(t)\|_{L^2}^2 \le E(u(t)) + R(u(t)) + C \le 2E(u(t)) + C.$$

In view that u conserves charge and energy we finally conclude that

$$\|u(t)\|_{H^1}^2 \le 4E(u(0)) + \|u(0)\|_{L^2}^2 + 2C,$$

which implies, together with

$$\lim_{t\to T^-} \|u(t)\|_{H^1} = +\infty, \quad \text{if } T < \infty,$$

that the time of existence of the solution u is $T = +\infty$.

□

Existence of standing waves

ODE problem

Recall the profile equation

$$\begin{cases} \phi'' + Z\delta(x)\phi + \omega\phi + f(|\phi|^2)\phi = 0, \\ \phi \in H^1(\mathbb{R}), \end{cases} \tag{ODE}$$

Hypothesis on f:

$$\begin{aligned} f \in C^1((0,+\infty);\mathbb{R}) \quad &\text{with } \ f(0) = 0, \\ f'(x) < 0 \quad &\text{for all } x > 0. \end{aligned} \tag{H$_f$}$$

$\phi \in H^1(\mathbb{R})$ is a solution in the distributional sense if for every $\chi \in H^1(\mathbb{R})$

$$\begin{aligned} 0 = \mathrm{Re}\Bigg[\int_{-\infty}^{+\infty} \phi'(x)\overline{\chi'(x)}dx - Z\phi(0)\overline{\chi(0)} - \omega\int_{-\infty}^{+\infty} \phi(x)\overline{\chi(x)}dx \\ - \int_{-\infty}^{+\infty} f(|\phi|^2(x))\phi(x)\overline{\chi(x)}dx\Bigg]. \end{aligned}$$

Analysis of the (ODE) (i)

Lemma

Let $\phi \in H^1(\mathbb{R})$, *with* $\phi'' + Z\delta(x)\phi + \omega\phi + f(|\phi(x)|^2)\phi(x) = 0$ *in the distributional sense, then*

$$\phi \in C^j(\mathbb{R}\backslash\{0\}) \cap C(\mathbb{R}), \quad j = 1,2. \tag{1a}$$

$$\phi''(x) + \omega\phi(x) + f(|\phi(x)|^2)\phi(x) = 0, \qquad \textit{for} \quad x \neq 0. \tag{1b}$$

$$\phi'(0+) - \phi'(0-) = -Z\phi(0). \tag{1c}$$

$$\phi'(x), \phi(x) \to 0, \qquad \textit{if} \quad |x| \to \infty. \tag{1d}$$

$$|\phi'(x)|^2 + \omega|\phi(x)|^2 + g(|\phi(x)|^2) = 0, \qquad \textit{for} \quad x \neq 0. \tag{1e}$$

$$\textit{where} \qquad g(s) = \int_0^s f(s)\, ds.$$

Analysis of the (ODE) (ii)

Lemma

Let $p > 1$, $\omega, \lambda_1, \lambda_2 \in \mathbb{R}$ and $Z \in \mathbb{R} \setminus \{0\}$. Let ϕ be a non-trivial solution to (1a) - (1e). *Then $\phi(x) \neq 0$ for all $x \in \mathbb{R}$ and $|\phi| > 0$. $-\phi$ is also a solution.*

Lemma (Useful)

Let $p > 1$, $\omega, \lambda_1, \lambda_2 \in \mathbb{R}$ and $Z \in \mathbb{R} \setminus \{0\}$. Let ϕ be a non-trivial solution to (1a) - (1e). *Then we have either one of the following:*

(i) *$Im(\phi(x)) = 0$ for all $x \in \mathbb{R}$; or,*

(ii) *there exists $c \in \mathbb{R}$ such that $Re(\phi(x)) = c\, Im(\phi(x))$ for all $x \in \mathbb{R}$.*

Explicit profile construction for $\omega \neq 0$, $Z = 0$, $\lambda_1 \leq 0$, $\lambda_2 < 0$

By using $\phi, \phi' \to 0$ as $x \to \infty$ we obtain

$$[\phi']^2 + \omega\phi^2 + 2\alpha\phi^{p+1} + \beta\phi^{2p} = 0,$$

with $\alpha = \lambda_1/(p+1)$, $\beta = \lambda_2/p$. Then,

$$\phi(x) = \left[-\frac{\alpha}{\omega} + \frac{\sqrt{\omega\beta - \alpha^2}}{\omega} \sinh\left((p-1)\sqrt{-\omega}x\right) \right]^{-\frac{1}{p-1}},$$

is the profile of the standing wave solution provided that

$$-\frac{p\lambda_1^2}{(p+1)^2\lambda_2} < -\omega.$$

Explicit construction

$$\text{The function} \quad \phi_1(x) := \phi(-|x| - d), \qquad -l < d,$$

satisfies all the properties of our first lemma except possibly the jump condition: $\phi'(0+) - \phi'(0+) = -Z\phi(0)$. If we consider $R_1 : (-l, \infty) \to (1, \infty)$ the diffeomorphism defined by

$$R_1(d) = \frac{\sqrt{\omega\beta - \alpha^2}\cosh((p-1)\sqrt{-\omega}d)}{\sqrt{\omega\beta - \alpha^2}\sinh((p-1)\sqrt{-\omega}d) + \alpha}.$$

then, we get

$$d = R_1^{-1}\left(\frac{Z}{2\sqrt{-\omega}}\right), \qquad \text{with} \qquad Z > 0 \;\text{ and }\; -\omega < \frac{Z^2}{4}.$$

Profile existence theorem; case $\omega \neq 0$

Theorem

Let $p > 1$, $\lambda_1 \leq 0$, $\lambda_2 < 0$ and $Z > 0$ in the NLS equation. Then for all values of $\omega < 0$ satisfying

$$-\frac{p\lambda_1^2}{(p+1)^2\lambda_2} < -\omega < \frac{Z^2}{4}$$

the familiy of standing wave solutions, $u(x,t) = e^{-i\omega t}\phi_\omega$, with ϕ_ω given by

$$\phi_\omega = \left[\frac{\alpha}{-\omega} + \frac{\sqrt{v}}{-\omega}\sinh\left((p-1)\sqrt{-\omega}\left(|x| + R_1^{-1}\left(\frac{Z}{2\sqrt{-\omega}}\right)\right)\right)\right]^{-\frac{1}{p-1}}$$

are solutions to the NLS equation. Here $v = \omega\beta - \alpha^2$.

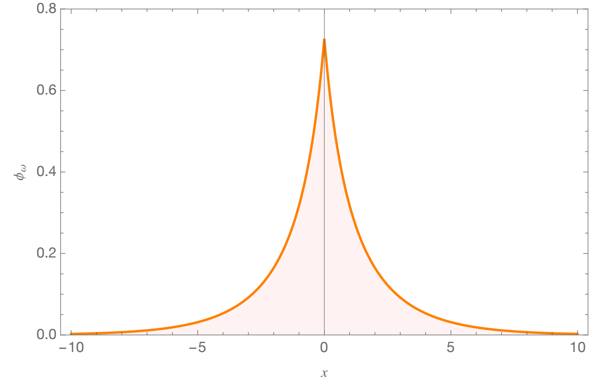

Figure 1: Profile function $\phi_\omega = \phi_\omega(x)$ for parameter values $\omega = -0.25$, $Z = 2$, $p = 3$, $\lambda_1 = \lambda_2 = -1$.

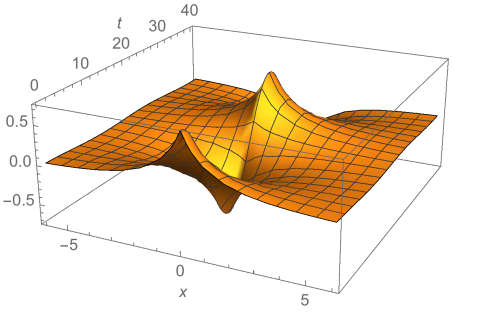

Figure 2: Time evolution of the standing wave solution $u(x,t) = e^{-i\omega t}\phi_\omega(x)$ with $\omega = -0.25$, $Z = 2$ and in the case of a quintic/cubic ($p = 3$), doubly repulsive ($\lambda_1 = \lambda_2 = -1$) nonlinearity.

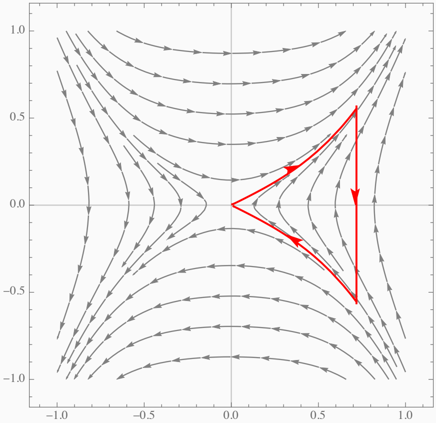

Figure 3: Dynamics in the (ϕ, ϕ')-plane for $f(x) = -x(1+x^2)$, that is, for $\lambda_1 = \lambda_2 = -1$ and $\omega = -0.25$, in the case of a quintic/cubic nonlinearity with $p = 3$.

Orbital stability

Critical points

Let us consider the functional $G_\omega : H^1(\mathbb{R}) \to \mathbb{R}$ for values $\omega \leq 0$, defined as

$$G_\omega(v) = \frac{1}{2}\|v_x\|_{L^2}^2 - \frac{Z}{2}|v(0)|^2 - \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{1}{2}\int_{-\infty}^{\infty} g(|v(x)|^2)dx,$$

and the set of critical points associated to G_ω as

$$\mathscr{A}_\omega = \{v \in H^1(\mathbb{R}) : G'_\omega(v) = 0, v \neq 0\}.$$

Here $g = g(\cdot)$ is the antiderivative of $f = f(\cdot)$. For $\phi \in \mathscr{A}_\omega$ we have the relation

$$G'_\omega(\phi) = A_Z\phi - \omega\phi - f(|\phi|^2)\phi$$

Properties of the set of critical points (i)

Lemma

Let $1 < p < \infty$*,* $Z > 0$ *and let* $\omega \in \mathbb{R}$ *be such that* $\omega + \frac{Z^2}{4} \leq 0$*. Then the set* $\mathscr{A}_\omega$ *is empty.*

Proof. If there exists $h \in H^1(\mathbb{R}) \setminus \{0\}$ satisfying $G'_\omega(h) = 0$, then

$$0 = \frac{d}{ds} G(sh)\Big|_{s=1}, \quad \text{and since} \quad \langle A_Z h, h\rangle \geq -\frac{Z^2}{4}\|h\|^2_{L^2}$$

for all $h \in H^1(\mathbb{R})$, we then obtain

$$\begin{aligned} 0 &= \|h_x\|^2_{L^2} - Z|h(0)|^2 - \omega\|h\|^2_{L^2} - \int_{-\infty}^{\infty} f(|h(x)|^2)|h(x)|^2 dx \\ &\geq -(Z^2/4 + \omega)\|h\|^2_{L^2} - \int_{-\infty}^{\infty} f(|h(x)|^2)|h(x)|^2 dx \\ &\geq -\int_{-\infty}^{\infty} f(|h(x)|^2)|h(x)|^2 dx > 0, \end{aligned}$$

□

Properties of the set of critical points (ii)

Lemma
Let $1 < p < \infty$ *and* $Z \in \mathbb{R}$. *If* $\omega > 0$ *then* $\mathscr{A}_\omega = \varnothing$.

Lemma
Let $\omega \in \mathbb{R}$ *and* $Z < 0$. *Then, we have that* $\mathscr{A}_\omega = \varnothing$.

Properties of the set of critical points (ii)

Lemma
Let $1 < p < \infty$ *and* $Z \in \mathbb{R}$*. If* $\omega > 0$ *then* $\mathscr{A}_\omega = \varnothing$*.*

Lemma
Let $\omega \in \mathbb{R}$ *and* $Z < 0$*. Then, we have that* $\mathscr{A}_\omega = \varnothing$*.*

Proofs by contradiction.

Properties of the set of critical points (iii)

Lemma

Let $p > 1$, $\lambda_1 < 0$, $\lambda_2 < 0$, $Z > 0$ *and* ω *such that* $-\frac{p\lambda_1^2}{(p+1)^2\lambda_2} < -\omega < \frac{Z^2}{4}$.
Considering $f(x) = \lambda_1 x^{(p-1)/2} + \lambda_2 x^{p-1}$, *then*

$$\mathscr{A}_\omega = \{e^{i\theta}\phi_\omega : \theta \in \mathbb{R}\}.$$

Proof. It is clear that for all $\theta \in \mathbb{R}$, $e^{i\theta}\phi_\omega \in \mathscr{A}_\omega$. Conversely, if $g \in \mathscr{A}_\omega$, then g satisfies all the necessary conditions to be a solution of the Euler-Lagrange equation and $|g| > 0$. Goal: to show that there exist $\theta \in \mathbb{R}$ such that $g(x) = e^{i\theta}\phi_\omega(x)$ for all $x \in \mathbb{R}$.

- $\phi_\omega \in D(A_Z)$ is the unique positive solution of the Euler-Lagrange equation. Indeed, if $v \in H^1(\mathbb{R})$ is a positive solution then v satisfies the IVP

$$\begin{cases} -\psi''(x) = \omega\psi(x) + f(\psi^2(x))\psi(x) := H(\psi(x)), \quad x > 0, \\ \quad \psi(0) = c_0, \quad \psi'(0) = -Zc_0/2, \end{cases}$$

where c_0 is the unique positive root of

$$\Phi_\omega(c, Zc/2) = \frac{Z^2}{4}c^2 + \omega c^2 + g(c^2).$$

Since H is locally Lipschitz around zero the IVP has a unique positive solution given by ϕ_ω. Thus, $v \equiv \phi_\omega$ on $(0,\infty)$. Similar arguments show that $v \equiv \phi_\omega$ on $(-\infty, 0)$. Hence, $v(x) = \phi_\omega(x)$ for all $x \in \mathbb{R}$.

- If $g(x) = e^{i\theta(x)}\rho(x)$ then $\theta, \rho > 0$ satisfy

$$\begin{cases} \theta''\rho + 2\theta'\rho' = 0, \quad x > 0, \\ -(\theta')^2\rho + \rho'' + \omega\rho + f(|\rho|^2)\rho = 0, \quad x > 0. \end{cases}$$

The first equation together with the boundedness of $|g'|$ imply that $g(x) = e^{i\theta_0}\rho(x)$ for all $x \in (0, +\infty)$. Then, from second equation and by the analysis above we necessarily have that $g(x) = e^{i\theta_0}\phi_\omega(x)$ for all $x \in (0, \infty)$. A similar analysis shows that $g(x) = e^{i\theta_1}\phi_\omega(x)$ for all $x \in (-\infty, 0)$. Hence,

$$g(x) = e^{i\theta_0}\phi_\omega(x), \qquad \text{for all } x \in \mathbb{R}.$$

□

The minimization problem

Let us suppose that $1 < p < \infty$, $Z > 0$, $\lambda_1 \leq 0$, $\lambda_2 < 0$, ω is such that

$$-\frac{p\lambda_1^2}{(p+1)^2\lambda_2} < -\omega < \frac{Z^2}{4}, \qquad \text{and} \qquad f(x) = \lambda_1 x^{(p-1)/2} + \lambda_2 x^{p-1}.$$

Minimization problem associated to G_ω:

$$m(\omega) = \inf\{G_\omega(v) \, : \, v \in H^1(\mathbb{R})\},$$

and the minimal set

$$M(\omega) = \{u \in H^1(\mathbb{R}) \, : \, G_\omega(u) = m(\omega)\}.$$

The set of minima

Lemma
$-\infty < m(\omega) < 0$ *and* $M(\omega) \subset \mathscr{A}_\omega$.

Proof. First verify that $-\infty < m(\omega)$. Write

$$G_\omega(v) = R(v) - \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{\lambda_1}{p+1}\|v\|_{L^{p+1}}^{p+1}, \qquad v \in H^1(\mathbb{R}).$$

Then, by the auxiliary bound lemma we get

$$G_\omega(v) \geq R(v) \geq \frac{Z}{2}|v(0)|^2 - C \geq -C,$$

for all $v \in H^1(\mathbb{R})$ and some uniform $C > 0$, yielding $-\infty < m(\omega)$.

To show that $m(\omega) < 0$, let $v(x) := sh(x) \in H^1(\mathbb{R})$ with $s > 0$ and where $h(x) = e^{-\frac{Z|x|}{2}}$ is the eigenfunction of the operator A_Z associated to the eigenvalue $\frac{-Z^2}{4}$. Therefore

$$G_\omega(v) = -\frac{s^2}{2}\Big(\frac{Z^2}{4} + \omega\Big)\|h\|_{L^2}^2 - \frac{1}{2}\int_{-\infty}^{\infty} g(s^2h^2(x))dx.$$

Since $-g(s^2h^2(x)) < -f(s^2)s^2h^2(x)$,

$$G_\omega(v) \leq -\frac{s^2}{2}\|h\|_{L^2}^2\Big(\frac{Z^2}{4} + \omega + f(s^2)\Big).$$

Since $Z^2/4 + \omega > 0$ and $\lim_{s\to 0^+} f(s^2) = 0$ we conclude that there exists $s_0 > 0$ such that $Z^2/4 + \omega > -f(s^2) > 0$ for $0 < s \leq s_0$ and so $G_\omega(s_0 h) < 0$. Lastly, suppose $M(\omega) \neq \varnothing$. Then since for $h \in M(\omega)$ we have $h \neq 0$ and $G'_\omega(h) = 0$, then by previous Lemmata we obtain $M(\omega) \subset \mathscr{A}_\omega$. □

Auxiliary result: Brézis-Lieb lemma

A refinement of Fatou's lemma:

Lemma (Brézis-Lieb, 1983)

Let $2 \leq q < \infty$ *and* $\{u_j\}$ *be a bounded sequence in* $L^q(\mathbb{R})$ *such that* $u_j(x) \to u(x)$ *a.e. in* $x \in \mathbb{R}$ *as* $j \to \infty$*. Then,*

$$\|u_j\|_{L^q}^q - \|u_j - u\|_{L^q}^q - \|u\|_{L^q}^q \to 0, \quad \text{as } j \to \infty.$$

Compactness

Lemma

Let $h_n \in H^1(\mathbb{R})$ *be such that* $\lim_{n\to\infty} G_\omega(h_n) = m(\omega)$. *Then there exists a subsequence* h_{n_j} *and* $h \in H^1(\mathbb{R})$ *such that* $\lim_{n_j\to\infty} h_{n_j} = h$ *in* $H^1(\mathbb{R})$ *and* $G_\omega(h) = m(\omega)$.

Proof. First, notice that for all $v \in H^1(\mathbb{R})$

$$\begin{aligned} I_\omega(v) &:= \frac{1}{2}\|v_x\|_{L^2}^2 - \frac{\omega}{2}\|v\|_{L^2}^2 \\ &= G_\omega(v) + \frac{Z}{2}|v(0)|^2 + \frac{\lambda_1}{p+1}\|v\|_{L^{p+1}}^{p+1} + \frac{\lambda_2}{2p}\|v\|_{L^{2p}}^{2p}. \end{aligned}$$

Since $\omega < 0$, it follows that $I_\omega(v)$ is equivalent to $\|v\|_{H^1}^2$. From the fact that $\lambda_1, \lambda_2 < 0$, we obtain

$$\frac{1}{2}\|v_x\|_{L^2}^2 - \frac{\omega}{2}\|v\|_{L^2}^2 \le G_\omega(v) + R(v) + C \le 2G_\omega(v) + C,$$

for some uniform $C > 0$.

Hence, it is clear that if the sequence $G_\omega(h_n)$ converges then the sequence h_n is bounded in $H^1(\mathbb{R})$. Thus, there exists a subsequence h_{n_j} and $h \in H^1(\mathbb{R})$ such that $\{h_{n_j}\}$ converges wealky to h in $H^1(\mathbb{R})$. Since $H^1(-1,1)$ is compactly embedded in $C[-1,1]$, we deduce that $h_{n_j}(0) \to h(0)$. Thus,

$$m(\omega) \leq G_\omega(h) \leq \liminf_{n_j \to \infty} G_\omega(h_{n_j}) = m(\omega),$$

which implies that $h \in M(\omega)$.

Now, since $h_{n_j} \rightharpoonup h$ weakly in $H^1(\mathbb{R})$ we have that $h_{n_j}(x) \to h(x)$ a.e. in $x \in \mathbb{R}$ and also that

$$\|h_{n_j} - h\|_{L^2}^2 + \|h\|_{L^2}^2 = \|h_{n_j}\|_{L^2}^2 + o(1),$$
$$\|\partial_x h_{n_j} - h_x\|_{L^2}^2 + \|h_x\|_{L^2}^2 = \|\partial_x h_{n_j}\|_{L^2}^2 + o(1),$$

as $n_j \to \infty$.

$\|h_{n_j}\|_{H^1}$ uniformly bounded $\Rightarrow$ $\|h_{n_j}\|_{L^{p+1}}$ and $\|h_{n_j}\|_{L^{2p}}$ are uniformly bounded (by Gagliardo-Nirenberg interpolation inequalities). As $h_{n_j}(x) \to h(x)$ a.e. in $x \in \mathbb{R}$, by Brézis-Lieb lemma we get

$$\|h_{n_j} - h\|_{L^{p+1}}^{p+1} + \|h\|_{L^{p+1}}^{p+1} = \|h_{n_j}\|_{L^{p+1}}^{p+1} + o(1),$$
$$\|h_{n_j} - h\|_{L^{2p}}^{2p} + \|h\|_{L^{2p}}^{2p} = \|h_{n_j}\|_{L^{2p}}^{2p} + o(1),$$

as $n_j \to \infty$.

Combining yields

$$G_\omega(h_{n_j} - h) + G_\omega(h) = G_\omega(h_{n_j}) + o(1), \qquad \text{as } n_j \to \infty.$$

From the def. of I_ω,

$$\begin{aligned} 0 \le I_\omega(h_{n_j} - h) &\le I_\omega(h_{n_j} - h) - \frac{\lambda_1}{p+1}\|h_{n_j} - h\|_{L^{p+1}}^{p+1} - \frac{\lambda_2}{2p}\|h_{n_j} - h\|_{L^{2p}}^{2p} \\ &= G_\omega(h_{n_j} - h) + \frac{Z}{2}|h_{n_j}(0) - h(0)|^2 \\ &= G_\omega(h_{n_j}) - G_\omega(h) + o(1), \end{aligned}$$

inasmuch as $h_{n_j}(0) \to h(0)$. This yields $h_{n_j} \to h$ in $H^1(\mathbb{R})$.

□

Characterization of the minimal set

Lemma
$M(\omega) = \mathscr{A}_\omega = \{e^{i\theta}\phi_\omega : \theta \in \mathbb{R}\}$, where ϕ_ω denotes the standing wave profile.

Proof. From the previous lemmas, we infer that $M(\omega) \neq \varnothing$. Then there exists $h \in H^1(\mathbb{R})$ such that $G_\omega(h) = m(\omega)$, that is, $h \in M(\omega)$. Since $M(\omega) \subset \mathscr{A}_\omega$, $h \in \mathscr{A}_\omega$. Thus, there exists $\theta_0 \in \mathbb{R}$ such that $h = e^{i\theta_0}\phi_\omega$. Now, since $\phi_\omega \in H^1(\mathbb{R})$ and

$$G_\omega(\phi_\omega) = G_\omega(h) = m(\omega),$$

then $\phi_\omega \in M(\omega)$. This implies that $\mathscr{A}_\omega \subset M(\omega)$. The other inclusion was already proved above.

□

Proof of the main theorem

Suppose that the standing wave $e^{-i\omega t}\phi_\omega$ is orbitally unstable. Then there exists $\varepsilon_0 > 0$, a sequence $\{h_n(t)\}$ of solutions of the NLS equation and a sequence $t_n > 0$, such that

$$\lim_{n\to\infty} \|h_n(0) - \phi_\omega\|_{H^1} = 0, \tag{2a}$$

$$\inf_{\theta\in\mathbb{R}} \|h_n(t_n) - e^{i\theta}\phi_\omega\|_{H^1} \geq \varepsilon_0. \tag{2b}$$

Since G_ω is conserved by the flow of the NLS equation, we get that $G_\omega(h_n(t_n)) = G_\omega(h_n(0))$ for all $n \in \mathbb{N}$. Then (2a) and continuity of G_ω yield

$$\lim_{n\to\infty} G_\omega(h_n(t_n)) = G_\omega(\phi_\omega) = m(\omega).$$

Henceforth, from the former results there exists a subsequence h_{n_j} such that $h_{n_j}(t_{n_j}) \to h$ with $G_\omega(h) = m(\omega)$. Then $h \in \mathscr{A}_\omega$ and $h = e^{i\theta_0}\phi_\omega$ for some $\theta_0 \in \mathbb{R}$. Therefore,

$$\lim_{n_j \to \infty} h_{n_j}(t_{n_j}) = e^{i\theta_0}\phi_\omega,$$

in $H^1(\mathbb{R})$, which contradicts

$$\inf_{\theta \in \mathbb{R}} \|h_n(t_n) - e^{i\theta}\phi_\omega\|_{H^1} \geq \varepsilon_0.$$

Hence, we conclude that $e^{-i\omega t}\phi_\omega$ is orbitally stable.

□

Reference

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Happy birthday Kevin!