Orbital stability of periodic traveling waves for nonlinear Klein-Gordon equations in several space dimensions

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Introduction

Motivation: the sine-Gordon equation

sine-Gordon equation in one dimension (laboratory coordinates):

$$u_{tt} - u_{xx} + \sin u = 0, \qquad x \in \mathbb{R}, \;\; t > 0,$$

Applications:

- Surfaces with negative Gaussian curvature (Eisenhart, 1909)
- Propagation of crystal dislocations (Frenkel and Kontorova, 1939)
- Elementary particles (Perring and Skyrme, 1962)
- Propagation of magnetic flux on a Josephson line (Scott, 1969)
- Dynamics of fermions in the Thirring model (Coleman, 1975)
- Oscillations of a rigid pendulum attached to a stretched rubber band (Drazin, 1983)

Superconductivity and quantum-tunneling

Josephson won the 1973 Nobel Prize in Physics for his discovery of the Josephson effect, describing the emergence of a supercurrent through a Josephson junction. The phase difference of wave functions of electrons in the super-conductors satisfy the sine-Gordon equation.

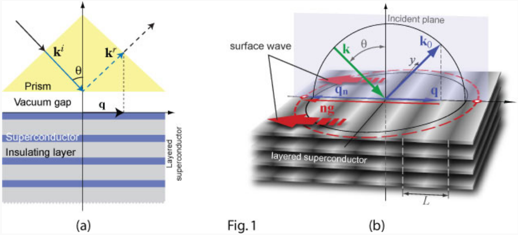

Figure 1: Two dimensional Josephson junction: infinite plates of superconductors separated by a thin dielectric barrier (image credit: AIST-NT, California, USA.)

The nonlinear Klein-Gordon equation

Nonlinear Klein-Gordon with periodic potential in 1D

$$u_{tt} - u_{xx} + V'(u) = 0.$$

for $(x,t) \in \mathbb{R} \times [0, +\infty)$, u scalar, $V \in C^2$, periodic. sine-Gordon: $V(u) = -\cos u$.

Assumptions on the potential:

(i) $V : \mathbb{R} \to \mathbb{R}$ is of class C^2 in all its domain and it is periodic with fundamental period P.

(ii) V has only non-degenerate critical points.

(iii) $V'(u)^4 (V(u)/V'(u)^2)'' \geq 0$ for all u under consideration.

Assumption (iii) implies monotonicity of the period map with respect to the energy.

Periodic traveling waves

$u(x,t) = \varphi(x - ct)$, $z = x - ct$, solution to the nonlinear pendulum equation:

$$(c^2 - 1)\varphi_{zz} + V'(\varphi) = 0,$$

$c \in \mathbb{R}$ (wave speed), $c^2 \neq 1$. Upon integration:

$$\tfrac{1}{2}(c^2 - 1)\varphi_z^2 = E - V(\varphi),$$

$E =$ constant (energy).

W.l.o.g.:

(iv) V has fundamental period $P = 2\pi$ and

$$\min_{u \in \mathbb{R}} V(u) = -1, \qquad \max_{u \in \mathbb{R}} V(u) = 1.$$

Classification

First dichotomy (wave speed c):

- Subluminal waves: $c^2 < 1$
- Superluminal waves: $c^2 > 1$

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- Subluminal waves: $c^2 < 1$
- Superluminal waves: $c^2 > 1$

Second dichotomy (energy E):

- Case $|E| < 1$, Librational wavetrain: $\varphi(z+T) = \varphi(z)$. Closed trajectory inside the separatrix in the phase portrait.
- Case $|E| > 1$, Rotational wavetrain: $\varphi(z+T) = \varphi(z) \pm 2\pi$. Open trajectory outside the separatrix in the phase plane. Sign φ_z is fixed. $E > 1$, superluminal case; $E < -1$, subluminal case.

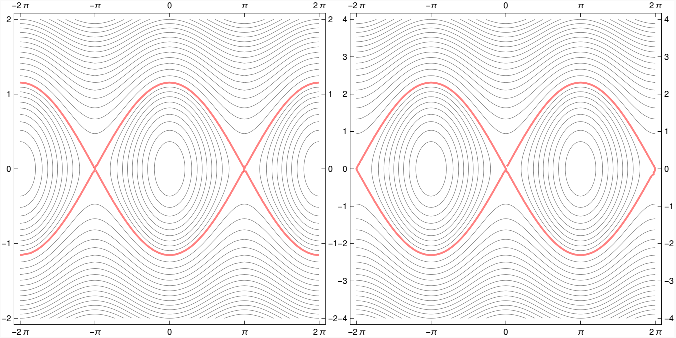

Figure 2: Phase portrait sine-Gordon case: $V(u) = 1 - \cos u$: superluminal $c^2 > 1$ (left); subluminal $c^2 < 1$ (right).

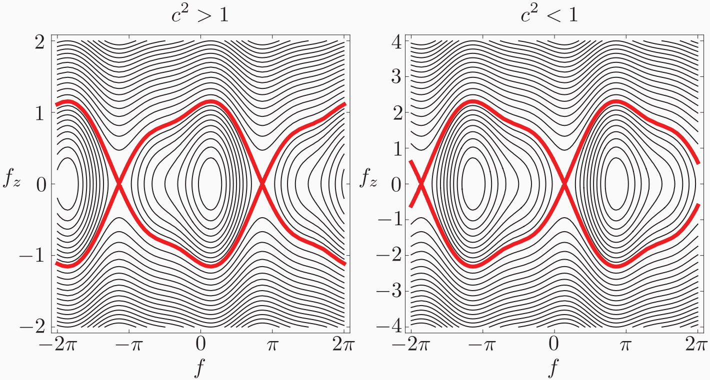

Figure 3: Phase portrait for $V(u) = -(0.861)(\cos u + \frac{1}{3}\sin(2u))$: superluminal $c^2 > 1$ (left); subluminal $c^2 < 1$ (right).

Subluminal rotations

Example: subluminal rotations for sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0,$$

"Periodic" wave, $u(x,t) = \varphi_{c,E}(x - ct)$, determined for $E < -1$, $c^2 < 1$ (subluminal rotation)

$$\varphi_{c,E}(z) = \begin{cases} -\arccos^{-1}\left[1 - 2\,\mathrm{cn}^{\,2}\left(\sqrt{\frac{1-E}{2(1-c^2)}}\,z; k\right)\right], & 0 \le z \le \frac{T}{2}, \\ \arccos^{-1}\left[1 - 2\,\mathrm{cn}^{\,2}\left(\sqrt{\frac{1-E}{2(1-c^2)}}\,(T - z); k\right)\right], & \frac{T}{2} \le z \le T, \end{cases}$$

$$k^2 = \frac{2}{1-E} \in (0,1), \qquad \text{elliptic modulus,}$$

$$\mathrm{cn} = \mathrm{cn}\,(\cdot), \qquad \text{elliptic cnoidal function}$$

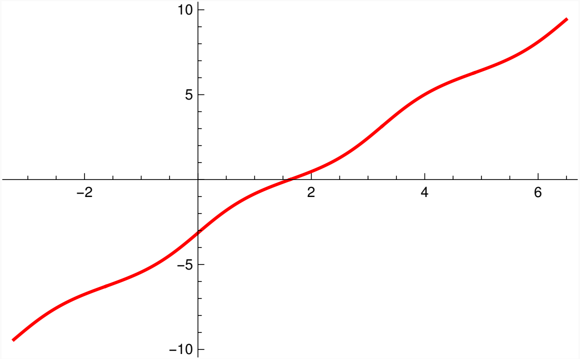

Figure 4: Rotational subluminal periodic wave $\varphi = \varphi_{c,E}(z)$ with $E = -2$, $c = 0.5$ in the interval $z \in [-T, 2T]$ where $T = 3.2476$.

Spectral stability

- Consider solutions of form $\varphi(z) + e^{\lambda t} w(z)$ (perturbation); $\lambda \in \mathbb{C}$.
- Linearize around the traveling wave φ to obtain equation for the perturbation

$$(c^2 - 1) w_{zz} - 2c\lambda w_z + \big(\lambda^2 + F'(\varphi(z))\big) w = 0 \qquad \text{(P)}$$

(quadratic pencil).

- Leads to associated spectral problem, definition of spectrum σ on $L^2(\mathbb{R};\mathbb{C})$. All σ is continuous (since coefficients are periodic).

Floquet spectrum

Parametrization the spectrum in terms of the Floquet multipliers $e^{i\theta} \in \mathbb{S}^1$, or $\theta \in \mathbb{R}$ (mod 2π). θ is the Floquet exponent. Let us define the set σ_θ as the set of complex numbers λ for which there exists $\theta \in \mathbb{R}$ and a nontrivial solution to (P) with quasi-periodic boundary conditions

$$w(T) = e^{i\theta} w(0).$$

Clearly $\sigma_\theta = \sigma_{\theta+2\pi k}$, for all $k \in \mathbb{Z}$. We thus define the Floquet spectrum σ_F as:

$$\sigma_F := \bigcup_{-\pi<\theta\leq\pi} \sigma_\theta$$

Floquet spectrum

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Theorem. $\sigma = \sigma_F$

Previous stability results for sine-Gordon

- **A.C. Scott**, Proc. IEEE (1969). Spectral stability.
- **G.B. Whitham**, *Linear and nonlinear waves* (1974). "Modulational" stability results. Based on modulation theory (Whitham, 1965).
- Forest, MacLaughlin (1982); Murakami (1986); Ercolani, Forest, McLaughlin (1990); Parkes (1991); etc. (abridged list).

Summary of stability results

Wave	Whitham (1974)	Scott (1969)
Subluminal rotational	stable	stable
Superluminal rotational	stable	unstable
Subluminal librational	unstable	unstable
Superluminal librational	unstable	unstable

Scott's results

Scott (1969):

$$y = \exp\left(\frac{-c\lambda z}{c^2-1}\right) w,$$

$$y_{zz} + \frac{V''(\varphi(z))}{c^2-1} y = \left(\frac{\lambda}{c^2-1}\right)^2 y =: \nu y. \qquad \text{(H)}$$

Hill's equation with period T. $\nu \in \sigma_H$ (Floquet spectrum of (H)) if there is a bounded solution y.

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Scott assumed that the transformation is isospectral: $(\sigma_H = \sigma)$. This is not true. Actually:

Lemma (Jones et al. (2013). If $\lambda \in \sigma_H \cap \sigma$ then $\lambda \in i\mathbb{R}$.

References:

- Jones, Marangell, Miller, P., Phys. D **251** (2013)
- Jones, Marangell, Miller, P., J. Differential Equations **257** (2014)
- Angulo, P., Stud. Appl. Math. **137** (2016)

Summary:

Jones et al. (2013)

- Correct proof of Scott's results (spectral)
- sine-Gordon case

Jones et al. (2014)

- More generic potentials
- Analysis of the monodromy map
- Modulational stability index
- Relation to Whitham's modulation theory

Angulo, P. (2016)

- Orbital (nonlinear) stability of subluminal rotational waves
- Multidimensional orbital stability

Numerical calculation of the Floquet spectrum for sine-Gordon

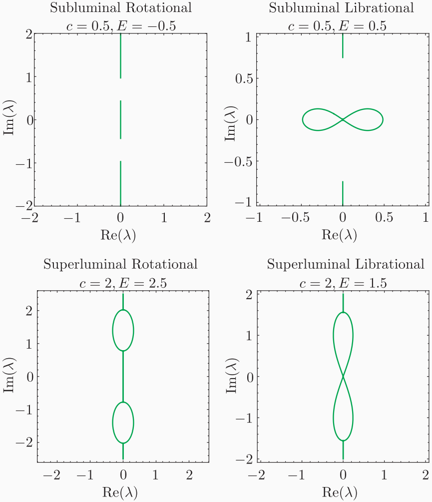

Figure 5: Numerical plots of the Floquet spectrum σ for sine-Gordon periodic wavetrains (Jones *et al.*, 2013)

Multidimensional nonlinear Klein-Gordon equation

Nonlinear Klein-Gordon equation in several space dimensions with periodic potential

$$u_{tt} - \Delta u + F(u) = 0, \qquad x \in \mathbb{R}^d,\ t > 0,$$

$d \geq 2$, $F(u) = V'(u)$, same assumptions on V. W.l.o.g. we assume $d = 2$.

Goal: Nonlinear (orbital) stability of the periodic subluminal rotational wave profile

$$\Phi(z, y) = \varphi(z), \quad (z, y) \in \mathbb{R}^2,$$

$z = x - ct$ under "generic" perturbations.

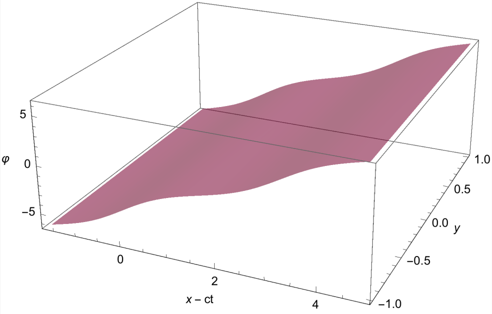

Figure 6: Rotational subluminal periodic wave $u(x,y,t) = \varphi_{c,E}(x-ct,y)$, parameter values $E=-2$, $c=0.5$ in the moving box $(x-ct,y)\in[-T/2,3T/2]\times[-1,1]$; here $T\approx 3.2476$.

Well-posedness theory

Preliminaries: periodic Sobolev spaces

$\mathscr{P} = C^\infty_{\text{per}}([0,T])$ - collection of functions $u : \mathbb{R} \to \mathbb{C}$ which are smooth and periodic with period $T > 0$. Topological dual $\mathscr{P}'$ - continuous linear functionals from $\mathscr{P}$ to $\mathbb{C}$ (set of periodic distributions).

$H^s_{\text{per}}([0,T])$, $s \in \mathbb{R}$, is the set of all $u \in \mathscr{P}'$ with

$$\|u\|^2_{H^s_{\text{per}}} = T \sum_{k\in\mathbb{Z}} (1+k^2)^s |\widehat{u}(k)|^2 < \infty.$$

We denote $H^0_{\text{per}}([0,T]) = L^2_{\text{per}}([0,T])$. Parseval: if $n \in \mathbb{N}$,

$$\|u\|^2_{H^n_{\text{per}}} = \sum_{j=0}^{n} \int_0^T |D^j_x u|^2 \, dx$$

Preliminaries: function spaces

Let us denote the Hilbert space

$$Y := H^1_{\mathrm{per}}([0,T]\times[0,L]) \times L^2_{\mathrm{per}}([0,T]\times[0,L]),$$

to represent perturbations which are square integrable, T-periodic in z and L-periodic in y, with $L>0$ arbitrary. The space Y is endowed by the standard norm

$$\|(u,v)\|_Y^2 = \|u\|^2_{H^1_{\mathrm{per}}} + \|v\|^2_{L^2_{\mathrm{per}}}, \qquad \text{for all } (u,v)\in Y,$$

where

$$\|u\|^2_{H^1_{\mathrm{per}}} = \|u_z\|^2_{L^2_{\mathrm{per}}} + \|u_y\|^2_{L^2_{\mathrm{per}}} + \|u\|^2_{L^2_{\mathrm{per}}}, \qquad \|u\|_{L^2_{\mathrm{per}}} = \int_0^T\int_0^L |u(z,y)|^2\,dy\,dz.$$

Standard inner product: $\langle\cdot,\cdot\rangle_Y$

Two dimensional nonlinear Klein-Gordon equation

Nonlinear Klein-Gordon in 2D

$$u_{tt} - u_{xx} - u_{yy} + F(u) = 0, \qquad \text{(nKG)}$$

$u = u(x, y, t)$, scalar, $(x, y) \in \mathbb{R}^2$ and $t \geq 0$. $F(u) = V'(u)$, periodic potential. Extrapolation to $d \geq 2$ is immediate.

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Theorem

The initial value problem associated to equation (nKG) *is globally well-posed in* Y.

Proof sketch (i)

- W.l.o.g. take $T = L = 2\pi$. Recast the equation as a first order system for a perturbation variable $v(z,y,t) = u(z,y,t) - \varphi(z)$, of form

$$\begin{aligned} \boldsymbol{v}_t &= L\boldsymbol{v} + R(\boldsymbol{v}), \quad (z,y,t) \in [0,2\pi]^2 \times (0,+\infty), \\ \boldsymbol{v}(0) &= \boldsymbol{v}_0, \qquad\qquad\qquad\qquad\qquad x \in [0,2\pi], \end{aligned}$$

where $\mathbf{v} = (v, v_t)^\top =: (v,w)^\top$, and

$$L = \begin{pmatrix} 0 & I \\ (1-c^2)\partial_z^2 + \partial_y^2 & 2c\partial_z \end{pmatrix}, \qquad R(\boldsymbol{v}) = \begin{pmatrix} 0 \\ F(\varphi) - F(\varphi + v) \end{pmatrix}.$$

- L is a linear, closed, densely defined operator in the Hilbert space $Y = H^1_{\mathrm{per}}([0,2\pi]\times[0,2\pi]) \times L^2_{\mathrm{per}}([0,2\pi]\times[0,2\pi])$, with dense domain $D(L) = H^2_{\mathrm{per}}([0,2\pi]\times[0,2\pi]) \times H^1_{\mathrm{per}}([0,2\pi]\times[0,2\pi])$

Proof sketch (ii)

- The operator $L : D(L) \subset Y \to Y$ is the infinitesimal generator of a C_0-group, $\{S(t)\}_{t\in\mathbb{R}}$ in Y. This fact can be verified via a direct computation of the group with standard Fourier analysis. Moreover, it can be shown that

$$\|S(t)(v_0, w_0)^\top\|_Y^2 \leqq 4\max\{1, t^2\}\|(v_0, w_0)^\top\|_Y^2,$$

for all $t > 0$, $(v_0, w_0)^\top \in Y$, as well as,

$$\begin{aligned}\|S(t)R(\mathbf{v}(s))\|_Y^2 &\leqq 4\max\{1, t^2\}\|(0, F(\varphi) - F(\varphi + v))^\top\|_Y^2 \\ &\leq 4\bar{C}\max\{1, t^2\}\|v(s)\|_{L^2}^2.\end{aligned}$$

Proof sketch (iii)

- Local well-posedness. The local existence of solutions is proved via a standard contraction mapping argument. Let T be such that $0 < T \leq 1$. Let us define

$$Y_{T,\beta} := \Big\{ \mathbf{v} \in C([0,T];Y) : \sup_{t\in[0,T]} \|\mathbf{v}(t)\|_Y < \beta \Big\},$$

and for fixed $\mathbf{v}_0 = (v_0, w_0)^\top \in Y$, the mapping

$$\Psi_{\mathbf{v}_0}(\mathbf{v})(t) := S(t)\mathbf{v}_0 + \int_0^t S(t-s)R(\mathbf{v}(s))\,ds.$$

We can choose $T > 0$ and $\beta > 0$ such that $\Psi_{\mathbf{v}_0}(\mathbf{v}(t)) \in Y_{T,\beta}$ for all $\mathbf{v} \in Y_{T,\beta}$ and that $\Psi_{\mathbf{v}_0}(\mathbf{v}(t)) : Y_{T,\beta} \to Y_{T,\beta}$ is a contraction.

Proof sketch (iv)

- Global well-posedness. Verify via *a priori* energy estimates, that the procedure above can be extended globally in time. If $\boldsymbol{v} = (v, w)^\top$ is a solution then

$$\begin{aligned} v_t &= w, \\ w_t &= (1 - c^2) v_{zz} + v_{yy} + 2c w_z + F(\varphi) - F(\varphi + v). \end{aligned}$$

Set

$$H(t) := \tfrac{1}{2} \left(\|v\|^2_{L^2_{\text{per}}} + (1 - c^2) \|v_z\|_{L^2_{\text{per}}} + \|v_y\|^2_{L^2_{\text{per}}} + \|w\|^2_{L^2_{\text{per}}} \right)$$

Upon integration by parts and periodicity

$$\frac{dH}{dt} = \int_0^{2\pi} \int_0^{2\pi} vw \, dz \, dy + \int_0^{2\pi} \int_0^{2\pi} F(\varphi) - F(\varphi + v) \, dz \, dy$$

Proof sketch (v)

$$\frac{dH}{dt} \leqq (1+\bar{C}) \int_0^{2\pi} \int_0^{2\pi} |v||w| \, dz \, dy \leqq C(\|v\|^2_{L^2_{\text{per}}} + \|w\|^2_{L^2_{\text{per}}}) \leqq CH(t),$$

for some uniform $C > 0$. Thus, by Gronwall's lemma we obtain

$$H(t) \leqq e^{Ct} H(0) \leqq C(T) H(0).$$

Hence, the solution can be extended globally in time by the same procedure. We conclude that there exists a unique global solution $\mathbf{v} \in C([0, +\infty); Y)$ to the Cauchy problem.

Orbital stability

Preliminaries

Interested in the dynamics of the set

$$\mathscr{O}_\varphi = \{\varphi(\cdot + \zeta) : \zeta \in \mathbb{R}\}$$

under the flow generated by (nKG). Consider the space

$$\mathscr{P}_\pm(T) := \{u : \mathbb{R} \to \mathbb{R} : u(z+T) = u(z) \mp 2\pi, \text{ for all } z \in \mathbb{R}\},$$

i.e. u produces a translation of the fundamental period of V after a period T.

Main theorem (i)

Theorem (transverse orbital stability)

The rotational subluminal traveling wave profile $\Phi(z,y) = \varphi(z)$*,* $(z,y) \in \mathbb{R}^2$*, is orbitally stable in* Y *by the flow generated by the two-dimensional nonlinear Klein-Gordon equation* (nKG) *in the following sense: for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for* $u_0 = u_0(\cdot,\cdot) \in \mathscr{P}_{\pm}(T) \times H^1_{\mathrm{per}}([0,L])$ *and* $u_1 \in L^2_{\mathrm{per}}([0,T] \times [0,L])$ *satisfying*

$$\|u_0 - \Phi\|_{H^1_{\mathrm{per}}([0,T]\times[0,L])} + \|c\partial_z u_0 + u_1\|_{L^2_{\mathrm{per}}([0,T]\times[0,L])} < \delta,$$

then the solution $u = u(z,y,t)$ *to* (nKG) *with initial conditions* $u(\cdot,\cdot,0) = u_0(\cdot,\cdot)$ *and* $u_t(\cdot,\cdot,0) = u_1(\cdot,\cdot)$ *satisfies, for all* $t \geq 0$*,*

$$\begin{cases} t \to u(\cdot + ct, \cdot, t) - \Phi(\cdot,\cdot) \in H^1_{\mathrm{per}}([0,T] \times [0,L]) \\ t \to c\partial_z u(\cdot + ct, y, t) + u_t(\cdot + ct, y, t) \in L^2_{\mathrm{per}}([0,T] \times [0,L]), \end{cases}$$

and, for all $t > 0$*.*

Main theorem (ii)

Theorem (transverse orbital stability - continued)
Moreover,

$$\|u(\cdot+\gamma,\cdot,t)-\Phi(\cdot,\cdot)\|_{H^1_{\mathrm{per}}([0,T]\times[0,L])}+ \\ +\|c\partial_z u(\cdot,\cdot,t)+u_t(\cdot,\cdot,t)\|_{L^2_{\mathrm{per}}([0,T]\times[0,L])}<\varepsilon.$$

Here the modulation parameter γ is given explicitly by $\gamma(t)=ct$. In addition, we have $t\in\mathbb{R}\to u(\cdot,y,t)\in\mathscr{P}_{\pm}(T)$, for all y fixed and all $t>0$.

Remark. The notation $u_0(\cdot,\cdot) \in \mathscr{P}_{\pm}(T) \times H^1_{\rm per}([0,L])$ means:

$$\begin{cases} z \to u_0(z,y) \in \mathscr{P}_{\pm}(T), \text{ for every } y \in \mathbb{R} \\ u(z,\cdot) \in H^1_{\rm per}([0,L]), \text{ for every } z \in \mathbb{R}. \end{cases}$$

Perturbation variables

For any solution $u = u(x, y, t)$ to (nKG), consider the perturbation variable

$$v(z, y, t) = u(z + ct, y, t) - \varphi(z).$$

Suppose $x \to u(x, \cdot, t) \in \mathscr{P}_{\pm}(T)$ and $y \to u(\cdot, y, t) \in L^2_{\mathrm{per}}([0, L])$ for all $t \in \mathbb{R}$, then v is a doubly-periodic function on $\mathbb{R}^2$,

$$\begin{aligned} v(z + T, y + L, t) &= u(z + T + ct, y + L, t) - \varphi(z + T) \\ &= u(z + ct, y, t) \mp 2\pi - \varphi(z) \pm 2\pi = v(z, y, t). \end{aligned}$$

v satisfies the nonlinear equation

$$v_{tt} - 2cv_{zt} + (c^2 - 1)v_{zz} - v_{yy} + F'(\varphi(z) + v) - F'(\varphi(z)) = 0.$$

Need to study the nonlinear stability of the trivial solution $v \equiv 0$.

First order Hamiltonian system

Recast nonlinear eq. for v as a first order Hamiltonian system

$$\boldsymbol{v}_t = J\mathscr{E}'(\boldsymbol{v}),$$

where $\boldsymbol{v} = (v, v_t) := (v, w)^\top$,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 2c\partial_z \end{pmatrix},$$

and $\mathscr{E}'$ is the derivative of the well-defined smooth functional

$$\mathscr{E} : H^1_{\mathrm{per}}([0,T]\times[0,L]) \times L^2_{\mathrm{per}}([0,T]\times[0,L]) \to \mathbb{R},$$
$$\mathscr{E}(v,w) = \frac{1}{2}\int_0^T\int_0^L (1-c^2)v_z^2 + v_y^2 + w^2 + 2G(v)\,dy\,dz,$$

with $G'(v(z,y)) = F(\varphi(z) + v(z,y)) - F(\varphi(z))$.

Properties of the functional $\mathscr{E}$ (i)

- J is a skew-adjoint operator with respect to the inner product in $L^2_{\text{per}}([0,T]\times[0,L])$.
- Since for z fixed,

$$G(s)=\int_0^s F(\varphi(z)+\tau)-F(\varphi(z))\,d\tau,$$

then $|G(s)|\leq \frac{1}{2}s^2$ and $\mathscr{E}$ is well defined,

$$|\mathscr{E}(v,w)|\leq \frac{1}{2}(1-c^2)\|v_z\|^2_{L^2_{\text{per}}}+\|v_y\|^2_{L^2_{\text{per}}}+\frac{1}{2}\|w\|^2_{L^2_{\text{per}}}.$$

- The Hamiltonian structure implies that $\mathscr{E}$ is a conservation law.
- Also,

$$\mathscr{E}'(v,w)=\begin{pmatrix} (c^2-1)\partial_z^2 v-\partial_y^2 v+G'(v) \\ w \end{pmatrix}.$$

$\mathscr{E}'(0,0)=0.$

Properties of the functional $\mathscr{E}$ (ii)

- Stability of $\boldsymbol{v} \equiv (0,0)$ in Y requires to study the self-adjoint operator

$$\mathscr{E}''(v,w) = \begin{pmatrix} (c^2-1)\partial_z^2 - \partial_y^2 + F'(\varphi(z)+v) \\ w \end{pmatrix} : Y \to Y,$$

evaluated at $(v,w) = (0,0)$.

Lemma (spectral analysis of $\mathscr{E}''(0,0)$)

We consider the linear self-adjoint operator $\mathscr{E}''(0,0): Y \to Y$ with dense domain $D = H^2_{\mathrm{per}}([0,T]\times[0,L]) \times L^2_{\mathrm{per}}([0,T]\times[0,L])$. Then the spectrum $\sigma = \sigma(\mathscr{E}''(0,0))$ of $\mathscr{E}''(0,0)$ is discrete, $\sigma = \{0, \mu_1, \mu_2, ...\}$, where

$$0 < \mu_1 \le \mu_2 \le \mu_3 \le \mu_4 \le \cdots$$

and $\ker \mathscr{E}''(0,0) = span\{(\varphi_z, 0)\}$. Moreover, there exists $\beta > 0$ such that for every $\boldsymbol{h} \in Y$ satisfying $\boldsymbol{h} \perp (\varphi_z, 0)^\top$

$$\langle \boldsymbol{h}, \mathscr{E}''(0,0)\boldsymbol{h}\rangle_Y \ge \beta \|\boldsymbol{h}\|_Y^2.$$

Proof (i)

Proof. $\mathscr{E}''(0,0)(\varphi_z,0)^\top = 0$ because $\partial_y \varphi(z) = 0$ and φ is a solution to the spectral equation with $\lambda = 0$. Let $\mu < 0$ be an eigenvalue for $\mathscr{E}''(0,0)$ with $(h,g)^\top \in H^2_{\text{per}}([0,T]\times[0,L]) \times L^2_{\text{per}}([0,T]\times[0,L])$ eigenfunction. Thus,

$$\begin{cases} \mathscr{L}_1 h := (c^2-1)\partial_z^2 h - \partial_y^2 h + F'(\varphi(z))h = \mu h \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad g = \mu g. \end{cases}$$

It follows that $\mathscr{L}_1 h_y = \mu h_y$. So, h and h_y are eigenfunctions of $\mathscr{L}_1$. Next, we see that h is a function only of the variable z, namely, $h(z,y) = A(z)$ for all $(z,y) \in \mathbb{R}^2$. W.l.o.g. suppose that $\mu = \inf \sigma(\mathscr{L}_1)$. From a classical result on d-dimensional Schrödinger operators (cf. Eastham, 1973), $d \geq 2$, μ is a simple eigenvalue for $\mathscr{L}_1$ with an eigenfunction that does not take the value zero in $[0,T]\times[0,L]$.

Proof (ii)

Thus, suppose that $h(z,y) > 0$ for every z, y. Then, there exists $\theta > 0$ such that $h_y(z,y) = \theta h(z,y)$ for every z, y. For z fixed define $j(y) = h(z,y)$, so that j satisfies the following boundary problem,

$$\begin{cases} j'(y) = \theta j(y) \\ j(0) = h(z,0) =: A(z). \end{cases}$$

Therefore,

$$j(y) = h(z,y) = A(z)e^{\theta y}, \qquad \text{for all } \; y.$$

Since h is periodic in the y-variable, $\theta = 0$. Therefore, $h(z,y) = A(z)$ for all z, y, and satisfies

$$\mathscr{L}_1 A(z) = [(c^2-1)\partial_z^2 + F(\varphi(z))]A(z) = \mu A(z), \qquad \mu < 0.$$

This is a contradiction with oscillation theory for Hill's operators (Magnus, Winkler, 1966): $\mathscr{L}_1$ is a Hill's type scalar operator in $L^2_{\mathrm{per}}([0,T])$, and zero is the first eigenvalue of $\mathscr{L}$ and it is simple, with eigenfunction φ_z.

Proof (iii)

Moreover, $\sigma(\mathscr{L}_1) = \{0, \gamma_1, \gamma_2, \ldots\}$, where

$$0 < \gamma_1 \leq \gamma_2 < \gamma_3 \leq \gamma_4 < \cdots.$$

Hence, $\mathscr{E}''(0,0)$ is a non-negative operator.

By the analysis above $\mathscr{L}_1$ has no negative eigenvalues. Moreover, $\mathscr{L}_1 G = 0$ with $G(z,y) = \varphi_z \in H^2_{\mathrm{per}}([0,T] \times [0,L])$ and $G(z,y) > 0$ for all z, y. Therefore, zero is an simple eigenvalue for $\mathscr{L}_1$, it which implies that $\ker \mathscr{E}''(0,0) = \mathrm{span}\{(\varphi_z, 0)^\top\}$. The proof of the inequality follows by integration by parts.

$\square$

Coerciveness

Lemma

There exist $C_0 > 0$ and $\varepsilon > 0$ such that

$$\mathscr{E}(\boldsymbol{h}) \geq C_0 \|\boldsymbol{h}\|_Y^2,$$

for all $\boldsymbol{h} \in B(0;\varepsilon) = \{\boldsymbol{h} \in Y : \|\boldsymbol{h}\|_Y < \varepsilon\}$.

Proof. Since $\mathscr{E}(0,0) = \mathscr{E}'(0,0) = 0$,

$$\mathscr{E}(\boldsymbol{h}) = \frac{1}{2}\langle \boldsymbol{h}, \mathscr{E}''(0,0)\boldsymbol{h}\rangle_Y + o(\|\boldsymbol{h}\|_Y^2),$$

for every $\boldsymbol{h} \in B(0;\varepsilon)$. Hence, from the spectral theorem above we get that, for every $\boldsymbol{h} \in Y$,

$$\begin{aligned} &\boldsymbol{h} = \gamma(\varphi_z, 0)^\top + \boldsymbol{h}^\perp, \qquad \boldsymbol{h}^\perp \perp (\varphi_z, 0)^\top, \\ &\langle \boldsymbol{h}, \mathscr{E}''(0,0)\boldsymbol{h}\rangle_Y = \langle \boldsymbol{h}^\perp, \mathscr{E}''(0,0)\boldsymbol{h}^\perp\rangle_Y \geq \beta \|\boldsymbol{h}^\perp\|_Y^2. \end{aligned}$$

Therefore, we obtain for ε sufficiently small, that

$$\mathscr{E}(\boldsymbol{h}) \geq \beta \|\boldsymbol{h}^{\perp}\|_Y^2 + o(\|\boldsymbol{h}\|_Y^2) \geq C_0 \|\boldsymbol{h}\|_Y^2,$$

for some $C_0 > 0$ and $\|\boldsymbol{h}\|_Y < \varepsilon$.

$\square$

Therefore, we obtain for ε sufficiently small, that

$$\mathscr{E}(\boldsymbol{h}) \geq \beta \|\boldsymbol{h}^{\perp}\|_Y^2 + o(\|\boldsymbol{h}\|_Y^2) \geq C_0 \|\boldsymbol{h}\|_Y^2,$$

for some $C_0 > 0$ and $\|\boldsymbol{h}\|_Y < \varepsilon$.

□

$\mathscr{E}$ is a local Lyapunov function for the flow of the PDE.

Orbital stability of the trivial solution

Theorem

The trivial solution $\boldsymbol{v} \equiv (0,0)$ *is orbitally stable in* Y *by the periodic flow generated by the evolution equation* (nKG)*. That is, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for* $\boldsymbol{v}_0 \in Y$*, and* $\|\boldsymbol{v}_0\|_Y < \delta$*, we have that the global solution* $\boldsymbol{v}(t)$ *of* (nKG) *with* $\boldsymbol{v}(0) = \boldsymbol{v}_0$ *satisfies* $\boldsymbol{v}(t) \in Y$ *and* $\|\boldsymbol{v}(t)\|_Y < \varepsilon$ *for all* $t \geq 0$*.*

Proof.

Suppose that $\boldsymbol{v} = (0,0)$ is Y-unstable. Then we can choose initial data $\boldsymbol{v}_k(0) \in Y$ with $\|\boldsymbol{v}_k(0)\|_Y < 1/k$ and $\varepsilon > 0$, such that

$$\sup_{t \geq 0} \|\boldsymbol{v}_k(t)\|_Y \geq \varepsilon,$$

where $\boldsymbol{v}_k(t)$ is the solution to (nKG) with initial datum $\boldsymbol{v}_k(0)$.

Now, by continuity in t, we can select the first time t_k such that $\|\boldsymbol{v}_k(t_k)\|_Y = \frac{\varepsilon}{2}$. Since $\mathscr{E}$ is continuous over Y and is a conservation law for (nKG), we get from coerciveness, that

$$0 \leftarrow \mathscr{E}(\boldsymbol{v}_k(0)) = \mathscr{E}(\boldsymbol{v}_k(t_k)) \geq C_0 \|\boldsymbol{v}_k(t_k)\|_Y^2,$$

as $k \to \infty$, which contradicts the sup condition. This finishes the proof.

$\square$

Proof of main theorem

From the relation $v(x,y,t) = u(z+ct,y,t) - \varphi(z)$ and from the assumptions

$$(u_0,u_1) \in Y \subset L^2_{\rm per}([0,T]\times[0,L]) \times L^2_{\rm per}([0,T]\times[0,L]),$$

we obtain

$$v(z,y,0) = u_0(z,y) - \varphi(z) \in H^1_{\rm per}([0,T]\times[0,L]),$$
$$v_t(z,y,0) = c\partial_z u(z,y,0) + u_t(z,y,0) = c\partial_z u_0 + u_1 \in L^2_{\rm per}([0,T]\times[0,L]).$$

Therefore, from the definition of the Y-norm and from

$$\|u_0 - \Phi\|_{H^1_{\rm per}([0,T]\times[0,L])} + \|c\partial_z u_0 + u_1\|_{L^2_{\rm per}([0,T]\times[0,L])} < \delta,$$

apply orbital stability of the trivial solution to obtain

$$\begin{cases} t \to u(\cdot + ct, \cdot, t) - \Phi(\cdot, \cdot) \in H^1_{\text{per}}([0, T] \times [0, L]) \\ t \to c\partial_z u(\cdot + ct, y, t) + u_t(\cdot + ct, y, t) \in L^2_{\text{per}}([0, T] \times [0, L]), \end{cases}$$

and

$$\begin{aligned} &\|u(\cdot + ct, \cdot, t) - \Phi(\cdot, \cdot)\|_{H^1_{\text{per}}([0,T]\times[0,L])} + \\ &\qquad + \|c\partial_z u(\cdot, \cdot, t) + u_t(\cdot, \cdot, t)\|_{L^2_{\text{per}}([0,T]\times[0,L])} < \varepsilon. \end{aligned}$$

This finishes the proof.

□

Extension

Remark. It follows immediately that rotational subluminal traveling wavetrain profiles

$$\Phi(z, y_1, y_2, ..., y_{d-1}) = \varphi(z), \qquad (z, y_1, y_2, ..., y_{d-1}) \in \mathbb{R}^d,$$

where $\varphi(\cdot)$ is the one-dimensional subluminal rotational profile, are also nonlinearly stable in $H^1_{\mathrm{per}}([0,T] \times [0,L_1] \times \cdots \times [0,L_{d-1}]) \times L^2_{\mathrm{per}}([0,T] \times [0,L_1] \times \cdots \times [0,L_{d-1}])$ for any chosen wavelengths $L_i > 0$, $1 \leqq i \leqq d-1$, by the flow of the d-dimensional nonlinear Klein-Gordon equation.

Discussion

Open problems

- The orbital (in)stability with respect to co-periodic perturbations of superluminal rotational and superluminal librational waves has not been established, not even in one dimension. (Detection of a co-periodic eigenvalue.)
- We attempted to show orbital stability under two-dimensional perturbations which are co-periodic in the variable of propagation, but localized (i.e. in $L^2(\mathbb{R})$) in the transverse direction. It can be shown that the corresponding operator $\mathscr{E}''(0,0)$ has not closed range and $\lambda = 0$ belongs to the essential spectrum, precluding the existence of a spectral gap.
- The orbital, nonlinear stability of subluminal rotations under localized perturbations in the direction of propagation is an open problem, even in one spatial dimension.

Thanks!