Existence and spectral instability of bounded periodic waves for viscous balance laws

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Viscous balance laws

General family of **viscous balance laws**:

$$u_t + f(u)_x = u_{xx} + g(u), \qquad \text{(VBL)}$$

where $u = u(x,t) \in \mathbb{R}$ and $x \in \mathbb{R},\ t > 0$.

- $f = f(u)$ - nonlinear flux function,
- $g = g(u)$ - balance or production term

Interpretation

- Viscous balance laws appear as regularizations of scalar conservation laws:

$$u_t + f(u)_x = 0.$$

- Balance laws introduce a production term

$$u_t + f(u)_x = g(u).$$

- Viscosity or diffusion appears naturally as a regularization (parabolic) term, e.g., Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = u_{xx}.$$

- Viscous balance laws describe the evolution of a density u of point particles which:
  - diffuse
  - get advected with speed $f'(u)$
  - react a a per cápita production rate $g(u)/u$
- Applications to:
  - roll waves
  - nozzle flow
  - combustion theory
- Scalar viscous balance laws are simplified models that combine these effects into one single equation.

Hypotheses (i)

Assumptions:

$$f \in C^4(\mathbb{R}). \tag{H$_1$}$$

$g \in C^3(\mathbb{R})$ is of logistic or Fisher-KPP type:

$$\begin{aligned} & g(0) = g(1) = 0, \\ & g'(0) > 0, \;\; g'(1) < 0, \\ & g(u) > 0, \quad \forall\, u \in (0,1), \\ & g(u) < 0, \quad \forall\, u \in (-\infty, 0). \end{aligned} \tag{H$_2$}$$

Hypotheses (ii)

There exists $u_* \in (-\infty, 0)$ such that

$$\int_{u_*}^{0} g(s)\,ds + \int_{0}^{1} g(s)\,ds = 0. \tag{H$_3$}$$

$$\overline{a}_0 := f'''(0) - \frac{f''(0)g''(0)}{\sqrt{g'(0)}} \neq 0, \qquad \text{(genericity condition)} \tag{H$_4$}$$

Hypotheses (iii)

Under (H_2) and (H_3), $u_* \in (-\infty, 0)$ is the unique value such that (H_3) holds and

$$\int_u^1 g(s)\, ds > 0, \qquad \forall\, u \in (u_*, 1).$$

Therefore we can define

$$\gamma(u) := \sqrt{2 \int_u^1 g(s)\, ds}, \qquad u \in (u_*, 1),$$

as well as

$$I_0 := \int_{u_*}^1 \gamma(s)\, ds > 0, \qquad I_1 := \int_{u_*}^1 f'(s)\gamma(s)\, ds,$$

$$J := 2\int_{u_*}^1 f'(s)\sqrt{1+\gamma'(s)^2}\, ds, \quad L := 2\int_{u_*}^1 \sqrt{1+\gamma'(s)^2}\, ds.$$

Hypotheses (iv)

$$I_0 J \neq L I_1, \qquad \text{(non-degeneracy condition)} \qquad (\mathrm{H}_5)$$

$$f'(1) \neq \frac{I_1}{I_0}, \qquad \text{(saddle condition)} \qquad (\mathrm{H}_6)$$

Example

Burgers-Fisher equation:

$$u_t + uu_x = u_{xx} + u(1-u), \qquad x \in \mathbb{R},\ t > 0,$$

Burgers' flux:

$$f(u) = \frac{1}{2}u^2,$$

Logistic or Fisher-KPP reaction:

$$g(u) = u(1-u).$$

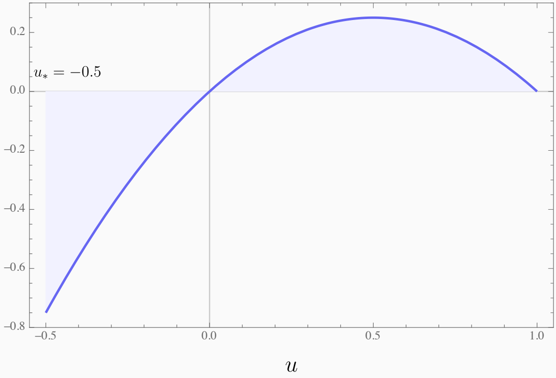

Figure: Logistic reaction function $g(u) = u(1-u)$.

Periodic traveling waves

A spatially periodic traveling wave is a solution of the form

$$u(x,t) = \varphi(x - ct),$$

where $c \in \mathbb{R}$ - wave speed, and $\varphi \in C^1(\mathbb{R})$ - profile function.

The wave is periodic with fundamental period $T > 0$ if

$$\varphi(z + T) = \varphi(z), \qquad \forall z \in \mathbb{R}, \quad z := x - ct,$$

The wave is bounded if

$$|\varphi(z)|, |\varphi'(z)| \leq C, \quad \forall z \in \mathbb{R}, \qquad \text{some } \; C > 0.$$

Theorem (existence of small amplitude periodic waves)

Under (H_1) *thru* (H_4), *there exist a critical speed* $c_0 := f'(0)$, *and* $\varepsilon_0 > 0$ *such that, for each* $0 < \varepsilon < \varepsilon_0$ *there exists a unique (up to translations) periodic traveling wave to* (VBL) *of the form* $u(x,t) = \varphi^\varepsilon(x - c(\varepsilon)t)$, *traveling with speed* $c(\varepsilon) = c_0 + \varepsilon$ *if* $\bar{a}_0 > 0$, *or* $c(\varepsilon) = c_0 - \varepsilon$ *if* $\bar{a}_0 < 0$, *with fundamental period,*

$$T_\varepsilon = \frac{2\pi}{\sqrt{g'(0)}} + O(\varepsilon), \qquad \text{as } \varepsilon \to 0^+.$$

The profile $\varphi^\varepsilon = \varphi^\varepsilon(\cdot)$ *is of class* $C^3(\mathbb{R})$, *satisfies* $\varphi^\varepsilon(z + T_\varepsilon) = \varphi^\varepsilon(z)$ *for all* $z \in \mathbb{R}$ *and is of small amplitude,*

$$|\varphi^\varepsilon(z)|, |(\varphi^\varepsilon)'(z)| \leq C\sqrt{\varepsilon},$$

for all $z \in \mathbb{R}$ *and some uniform* $C > 0$.

Theorem (existence of large period waves)

Under (H_1) - (H_3), (H_5) *and* (H_6), *there is a critical speed* $c_1 := I_1/I_0$ *such that the equation* (VBL) *has a traveling pulse solution (homoclinic orbit),* $u(x,t) = \varphi^0(x - c_1 t)$, *with speed* c_1, $\varphi^0 \in C^3(\mathbb{R})$ *and*

$$|\varphi^0(z) - 1|, |(\varphi^0)'(z)| \le Ce^{-\kappa|z|},$$

for all $z \in \mathbb{R}$, *some* $\kappa > 0$. *Moreover, there exists* $\varepsilon_1 > 0$ *such that, for each* $0 < \varepsilon < \varepsilon_1$ *there exists a unique periodic traveling wave solution to* (VBL) *of the form* $u(x,t) = \varphi^\varepsilon(x - c(\varepsilon)t)$, *with speed* $c(\varepsilon) = c_1 + \varepsilon$ *if* $f'(1) < c_1$ *or* $c(\varepsilon) = c_1 - \varepsilon$ *if* $f'(1) > c_1$, *with large fundamental period and bounded amplitude,*

$$T_\varepsilon = O(|\log \varepsilon|) \to \infty, \qquad |\varphi^\varepsilon(z)|, |(\varphi^\varepsilon)'(z)| = O(1),$$

as $\varepsilon \to 0^+$.

Theorem (existence of large period waves (continuation))

Moreover, the periodic orbits converge to the homoclinic (or traveling pulse) as $\varepsilon \to 0^+$ and satisfy (after a suitable reparametrization of z),

$$\sup_{z\in[-\frac{T_\varepsilon}{2},\frac{T_\varepsilon}{2}]} \big(|\varphi^0(z)-\varphi^\varepsilon(z)|+|(\varphi^0)'(z)-(\varphi^\varepsilon)'(z)|\big) \leq C\exp\Big(-\kappa\frac{T_\varepsilon}{2}\Big),$$

$$|c_1-c(\varepsilon)| = \varepsilon \leq C\exp\big(-\kappa T_\varepsilon\big),$$

for some uniform $C>0$, same $\kappa>0$ and for all $0<\varepsilon<\varepsilon_1$.

Theorem (spectral instability of small-amplitude waves)

Under conditions (H_1) *thru* (H_4), *there exists* $0 < \bar{\varepsilon}_0 < \varepsilon_0$ *such that every small-amplitude periodic wave* φ^ε *with* $0 < \varepsilon < \bar{\varepsilon}_0$ *is spectrally unstable: the Floquet spectrum of the linearized operator around the wave intersects the unstable half plane* $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathit{Re}\,\lambda > 0\}$.

Theorem (spectral instability of large period waves)

Under assumptions (H_1) - (H_3), (H_5) *and* (H_6), *there exists* $0 < \bar{\varepsilon}_1 < \varepsilon_1$ *such that every large period wave* φ^ε *with* $0 < \varepsilon < \bar{\varepsilon}_1$ *is spectrally unstable: the Floquet spectrum of the linearized operator around the wave intersects the unstable half plane* $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathit{Re}\,\lambda > 0\}$.

Existence of small amplitude periodic waves

Associated ODE system

Substitute $u(x,t) = \varphi(x - ct)$ into (VBL):

$$-c\varphi' + f'(\varphi)\varphi' = \varphi'' + g(\varphi), \qquad ' = d/dz.$$

Denote $U := \varphi(z)$, $V := \varphi'(z)$ to obtain the first order ODE planar system:

$$\begin{aligned} U' &= F(U,V,c) := V, \\ V' &= G(U,V,c) := -cV + f'(U)V - g(U). \end{aligned} \tag{ODE}$$

(H_1) and (H_2) imply: $F, G \in C^3(\mathbb{R}^3)$ and there exist two equilibrium points:

$$P_0 = (0,0), \qquad P_1 = (1,0).$$

Linearization at equilibria (i)

Jacobian with respect to (U, V) of the right hand side of (ODE):

$$A(U,V) := \begin{pmatrix} F_U & F_V \\ G_U & G_V \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ f''(U)V - g'(U) & -c + f'(U) \end{pmatrix}.$$

$A_0 = A(0,0)$ and $A_1 = A(1,0)$ are the linearizations of (ODE) evaluated at P_0 and P_1:

$$A_0 = \begin{pmatrix} 0 & 1 \\ -g'(0) & -c + f'(0) \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 1 \\ -g'(1) & -c + f'(1) \end{pmatrix}.$$

Linearization at equilibria (ii)

- Eigenvalues of A_1:

$$\lambda_1^{\pm}(c) = \frac{1}{2}\big(f'(1) - c\big) \pm \frac{1}{2}\sqrt{(f'(1)-c)^2 - 4g'(1)},$$

From (H$_2$), $g'(1) < 0$ and $P_1 = (1,0)$ is a hyperbolic saddle for each value of $c \in \mathbb{R}$.

- Eigenvalues of A_0:

$$\lambda_0^{\pm}(c) = \frac{1}{2}\big(f'(0) - c\big) \pm \frac{1}{2}\Big((f'(0)-c)^2 - 4g'(0)\Big)^{1/2}.$$

Hence $P_0 = (0,0)$ is a node, a focus or a center, depending on $c \in \mathbb{R}$.

$c \in \mathbb{R}$ is the bifurcation parameter.

Andronov-Hopf bifurcation (i)

Planar system of the form:

$$U' = F(U, V, \mu)$$
$$V' = G(U, V, \mu)$$

F, G of class C^3, $\mu \in \mathbb{R}$ - bifurcation parameter. Assume $P_0 = (U_0, V_0)$ is an equilibrium point; eigenvalues of the linearization at P_0: $\lambda^{\pm}(\mu) = \alpha(\mu) \pm i\beta(\mu)$. Assume that for $\mu = \mu_0$ the following conditions hold:

(a) non-hyperbolicity condition: $\alpha(\mu_0) = 0$, $\beta(\mu_0) = \omega_0 \neq 0$, and

$$\operatorname{sgn}(\omega_0) = \operatorname{sgn}((\partial G/\partial U)(U_0, V_0, \mu_0)).$$

(b) transversality condition:

$$\frac{d\alpha}{d\mu}(\mu_0) = d_0 \neq 0.$$

Andronov-Hopf bifurcation (ii)

(c) genericity condition: $a_0 \neq 0$, where a_0 is the first Lyapunov exponent,

$$a_0 := \frac{1}{16}\left(F_{UUU} + F_{UVV} + G_{UUV} + G_{VVV}\right) + \\ + \frac{1}{16\omega_0}\left(F_{UV}(F_{UU} + F_{VV}) - G_{UV}(G_{UU} + G_{VV}) - F_{UU}G_{UU} + F_{VV}G_{VV}\right),$$

(derivatives of F and G are evaluated at (U_0, V_0, μ_0)).

Then there exists $\varepsilon > 0$ such that a unique family of closed periodic orbit solutions bifurcates from the equilibrium point into the region:

$$\begin{cases} (\mu_0, \mu_0 + \varepsilon), & \text{if } a_0 d_0 < 0, \\ (\mu_0 - \varepsilon, \mu_0), & \text{if } a_0 d_0 > 0. \end{cases}$$

Andronov-Hopf bifurcation (iii)

Their amplitude and fundamental period behave like

$$|\varphi|, |\varphi_z| = O(\sqrt{|\mu - \mu_0|}), \quad T(\mu) = \frac{2\pi}{|\omega_0|} + O(|\mu - \mu_0|), \quad \text{as } \mu \to \mu_0.$$

Stability as solutions to the ODE. Orbits are:

stable (supercritical Hopf bifurcation) if $a_0 < 0$

unstable (subcritical Hopf bifurcation) if $a_0 > 0$.

Existence of small-amplitude periodic waves (i)

Proof of Theorem.

Write eigenvalues of A_0 as $\lambda_0^{\pm} = \alpha(c) \mp i\beta(c)$, where

$$\alpha(c) := \frac{1}{2}(f'(0) - c), \quad \beta(c) := -\frac{1}{2}\sqrt{4g'(0) - (f'(0) - c)^2}, \qquad c \approx f'(0).$$

Thus, $\alpha(c_0) = 0$ for the only bifurcation value of the speed: $c_0 := f'(0)$. At $c = c_0$ the origin is a center with eigenvalues

$$\lambda_0^{+}(c_0) = -i\sqrt{g'(0)}, \qquad \lambda_0^{-}(c_0) = i\sqrt{g'(0)}.$$

Notice that $\omega_0 := \beta(c_0) = -\sqrt{g'(0)} \neq 0$. Since $G_U = f''(U)V - g'(U)$,

$$(G_U)|_{(0,0,c_0)} = -g'(0) < 0,$$

yielding $\operatorname{sgn}(\omega_0) = \operatorname{sgn}((G_U)|_{(0,0,c_0)}) = -1$, that is, the non-hyperbolicity condition (a).

The transversality condition (b) also holds:

$$\frac{d\alpha}{dc}(c_0) = -\frac{1}{2} =: d_0 < 0.$$

Existence of small-amplitude periodic waves (ii)

The Lyapunov exponent reduces to

$$a_0 = \frac{1}{16}\big(G_{UUV} + G_{VVV}\big)|_{(0,0,c_0)} - \frac{1}{16\omega_0}\big(G_{UV}(G_{UU} + G_{VV})\big)|_{(0,0,c_0)}.$$

Upon calculations,

$$\begin{aligned} G_{UV}|_{(0,0,c_0)} = f''(0), \quad G_{UU}|_{(0,0,c_0)} &= -g''(0), \quad G_{VV}|_{(0,0,c_0)} = 0, \\ G_{UUV}|_{(0,0,c_0)} &= f'''(0), \quad G_{VVV}|_{(0,0,c_0)} = 0, \end{aligned}$$

we arrive at

$$a_0 = \frac{1}{16}\left(f'''(0) - \frac{f''(0)g''(0)}{\sqrt{g'(0)}}\right) = \frac{\bar{a}_0}{16} \neq 0,$$

in view of (H_4). This verifies the genericity condition (c). Since $d_0 < 0$ and $\mathrm{sgn}(a_0) = \mathrm{sgn}(\bar{a}_0)$ we obtain the result. $\square$

Examples

(I) Burgers-Fisher equation

The Burgers-Fisher equation:

$$u_t + uu_x = u_{xx} + u(1-u), \qquad x \in \mathbb{R},\ t > 0,$$

Burgers' flux:

$$f(u) = \frac{1}{2}u^2,$$

Logistic or Fisher-KPP reaction:

$$g(u) = u(1-u).$$

f and g satisfy (H_1) and (H_2). Here $u_* = -1/2$ is such that

$$\int_{u_*}^{1} g(s)\,ds = \int_{-\frac{1}{2}}^{1} s(1-s)\,ds = 0.$$

Thus, (H_3) is also satisfied.

Since $g'(u) = 1 - 2u$, $g''(u) = -2$, $f'(u) = u$, $f''(u) = 1$,

$$\overline{a}_0 = -\frac{f''(0)g''(0)}{\sqrt{g'(0)}} = 2 > 0,$$

Thus, the genericity condition (H_4) holds.

Bifurcation speed: $c_0 = f'(0) = 0$. Since $\overline{a}_0 > 0$ then for each speed value $c \in (0, \varepsilon_0)$ with $0 < \varepsilon_0 \ll 1$ there is a small amplitude periodic wave. This corresponds to a subcritical Hopf bifurcation. Their fundamental period is $T = 2\pi + O(c)$, for $c \approx 0^+$.

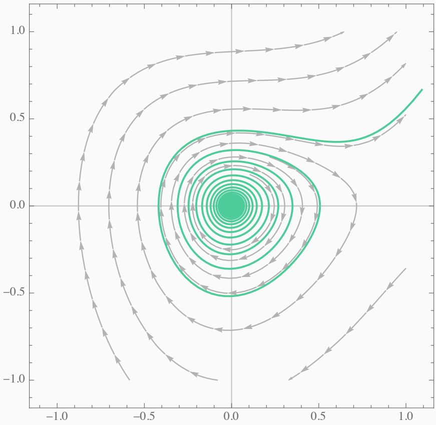

Figure: Phase portrait for the speed value $c = -0.05 < 0$. The origin is a repulsive node and all nearby solutions move away from it.

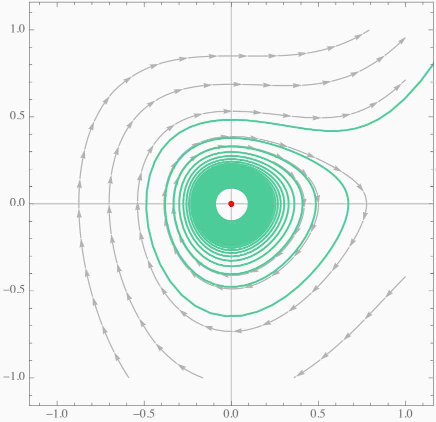

Figure: Phase portrait when $c = c_0 = 0$; a subcritical Hopf bifurcation occurs. The origin is a center and solutions move away if they start far enough from the origin and locally rotate around a linearized center otherwise.

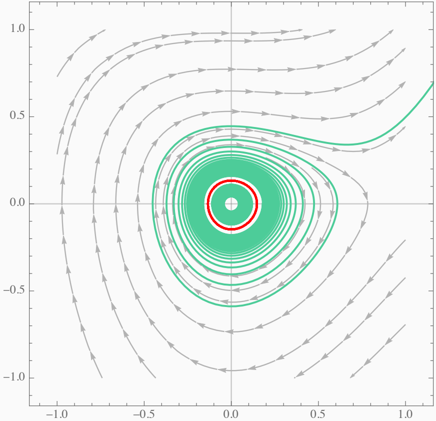

Figure: Phase portrait for $c = 0.005$. The orbit in red is a numerical approximation of the unique small amplitude periodic wave for this speed value, the origin is an attractive node and nearby solutions inside the periodic orbit approach zero, whereas solutions outside the periodic orbit move away since the orbit is unstable.

Figure: Graph (in red) of the periodic profile φ for $c = 0.005$ as a function of $z = x - ct$

Examples

(II) Logistic Buckley-Leverett model

Consider the following viscous balance law

$$u_t + \partial_x \left(\frac{u^2}{u^2 + \frac{1}{2}(1-u)^2} \right) = u_{xx} + u(1-u), \qquad x \in \mathbb{R}, \;\; t > 0,$$

underlies the nonlinear Buckley-Leverett flux function,

$$f(u) = \frac{u^2}{u^2 + \frac{1}{2}(1-u)^2}.$$

It captures the main features of two phase fluid flow in a porous medium. Again, logistic reaction: $g(u) = u(1-u)$.

f and g satisfy (H_1), (H_2) and (H_3). After computing the derivatives,

$$\bar{a}_0 = f'''(0) - \frac{f''(0)g''(0)}{\sqrt{g'(0)}} = 32,$$

and the genericity condition (H_4) holds.

Bifurcation speed: $c_0 = f'(0) = 0$. Since $\overline{a}_0 > 0$ then for each speed value $c \in (0, \varepsilon_0)$ with $0 < \varepsilon_0 \ll 1$ there is a small amplitude periodic wave. This corresponds to a subcritical Hopf bifurcation. Their fundamental period is $T = 2\pi + O(c)$, for $c \approx 0^+$.

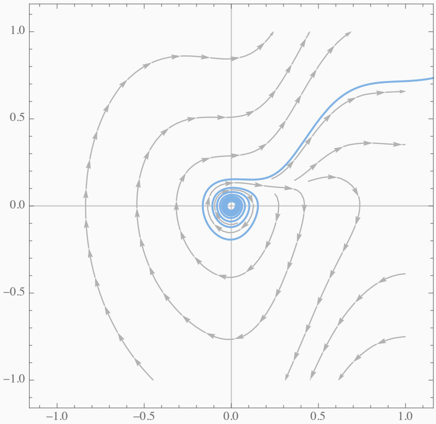

Figure: Phase portrait for the speed value $c = -0.05 < 0$. The origin is a repulsive node and all nearby solutions move away from it.

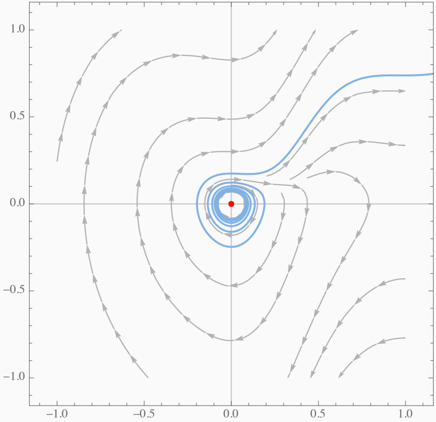

Figure: Phase portrait when $c = c_0 = 0$; a subcritical Hopf bifurcation occurs. The origin is a center and solutions move away if they start far enough from the origin and locally rotate around a linearized center otherwise.

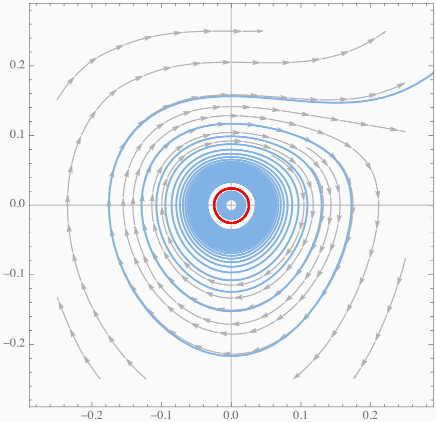

Figure: Phase portrait for $c = 0.0025$. The orbit in red is a numerical approximation of the unique small amplitude periodic wave for this speed value, the origin is an attractive node and nearby solutions inside the periodic orbit approach zero, whereas solutions outside the periodic orbit move away since the orbit is unstable.

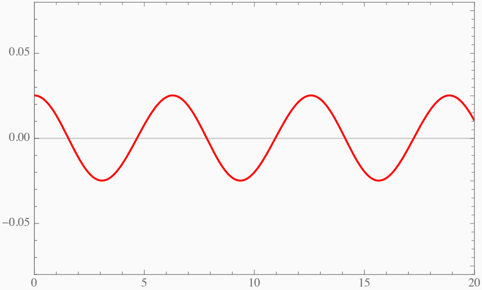

Figure: Graph (in red) of the periodic profile φ for $c = 0.0025$ as a function of $z = x - ct$

Examples

(III) Modified generalized Burgers-Fisher equation

Here

$$f(u) = \frac{1}{4}u^4 - \frac{1}{3}u^3,$$

$$g(u) = u - u^4.$$

Clearly, assumptions (H_1) - (H_3) hold, where the unique value $u_* \approx -0.72212$ is approximated numerically. Upon calculation of the derivatives,

$$\overline{a}_0 = -2 < 0.$$

Thus, (H_4) holds.

Bifurcation speed: $c_0 = f'(0) = 0$. Since $\overline{a}_0 < 0$ then the family occurs for negative speed values $c(\varepsilon) = -\varepsilon < 0 = c_0 = f'(0)$, sufficiently small. This is a supercritical Hopf bifurcation and the small amplitude periodic orbits are stable as solutions to the ODE, with speed value $c(\varepsilon) = -\varepsilon \in (-\varepsilon_0, 0)$ with $0 < \varepsilon_0 \ll 1$. Their fundamental period is $T = 2\pi + O(c)$, for $c \approx 0^+$.

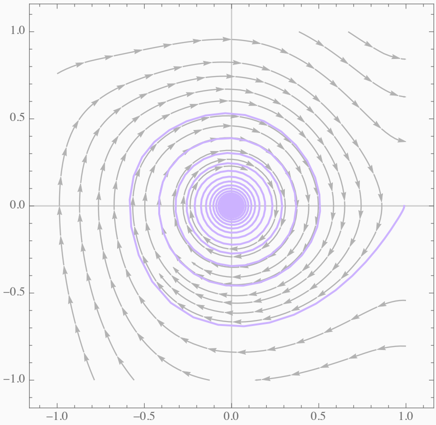

Figure: Phase portrait for the speed value $c = 0.05 > c_0 = 0$. The origin is an attractive node and all nearby solutions converge to it.

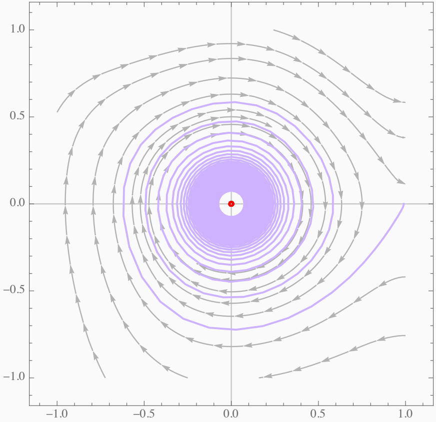

Figure: Phase portrait when $c = c_0 = 0$; a supercritical Hopf bifurcation occurs. The origin is a center and solutions move away if they start sufficiently far from the origin and rotate locally around a linearized center otherwise.

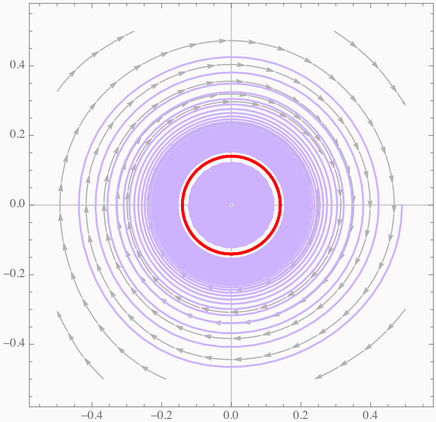

Figure: Phase portrait for $c = -0.005$. The orbit in red is a numerical approximation of the unique small amplitude periodic wave for this speed value, the origin is a repulsive node and nearby solutions both inside and outside the periodic orbit approach the periodic wave because it is stable as a solution to the ODE.

Figure: Graph (in red) of the periodic profile φ for $c = -0.005$ as a function of $z = x - ct$

Existence of large period waves

Augmented system

First we establish the existence of an homoclinic loop. We apply Melnikov's method. Consider the augmented system:

$$
\begin{aligned}
U' &= V, \\
V' &= -cV + af'(U)V - g(U),
\end{aligned}
\tag{A}
$$

where $a \in \mathbb{R}$ is a new auxiliary parameter. Write it in near-Hamiltonian form:

$$
\begin{aligned}
U' &= \partial_V H + \varepsilon R(U, V, \mu), \\
V' &= -\partial_U H + \varepsilon Q(U, V, \mu),
\end{aligned}
\tag{PS}
$$

where:

$$\mu = (\mu_1, \mu_2) \in \mathbb{R}^2, \quad a =: \varepsilon\mu_1,\ c =: \varepsilon\mu_2,$$

$$H(U,V) := \frac{1}{2}V^2 + \int_0^U g(s)\,ds,$$

is the Hamiltonian, and

$$R(U,V,\mu) \equiv 0,$$

$$Q(U,V,\mu) := \mu_1 f'(U)V - \mu_2 V,$$

are autonomous perturbations. Here $0 < \varepsilon \ll 1$ is small.

Hamiltionan system

The associated Hamiltonian (unperturbed) system is

$$\begin{aligned} U' &= \partial_V H = V, \\ V' &= -\partial_U H = -g(U). \end{aligned} \tag{HS}$$

Observations:

- $P_0 = (0,0)$ and $P_1 = (1,0)$ are equilibrium points for both the Hamiltonian system (HS) and the perturbed system (PS).
- It is easy to check that: P_0 is a center and P_1 is a hyperbolic saddle for the Hamiltonian system (HS).

- **Important**: $P_1 = (1,0)$ is also a hyperbolic saddle for the perturbed system (PS) for any parameter values a and c (equivalently, for any ε, μ_1 and μ_2). Indeed, the linearization of (PS) around $P_1 = (1,0)$ is

$$\widetilde{A}^{\varepsilon}(1,0) = \begin{pmatrix} 0 & -g'(1) \\ 1 & af'(1) - c \end{pmatrix},$$

having eigenvalues

$$\lambda_{\pm}^{\varepsilon} = \frac{1}{2}\left(af'(1) - c \pm \sqrt{(af'(1) - c)^2 - 4g'(1)}\right),$$

and in view of (H_2), we have $\lambda_{-}^{\varepsilon} < 0 < \lambda_{+}^{\varepsilon}$ for all values of a and c, yielding a hyperbolic saddle.

- $P_0 = (0,0)$ is a center for the perturbed system only if $c = af'(0)$ (which happens when $\varepsilon = 0$).

Energy levels for the Hamiltonian system (i)

- The energy levels at $P_0 = (0,0)$ and $P_1 = (1,0)$ as equilibria of the Hamiltonian system (HS) are

$$\beta := H(1,0) = \int_0^1 g(s)\,ds > 0,$$

and $H(0,0) = 0$.

- The set

$$\Gamma^\beta := \{(U,V) \in \mathbb{R}^2 : H(U,V) = \beta\},$$

is a homoclinic loop for the Hamiltonian system joining $P_1 = (1,0)$ with itself. It is given explicitly by

$$V(U) = \pm\overline{V}^\beta(U) := \pm\sqrt{2\Big(\beta - \int_0^U g(s)\,ds\Big)} = \pm\gamma(U), \quad U \in (u_*, 1).$$

Energy levels for the Hamiltonian system (ii)

- There exists a family of periodic orbits for the Hamiltonian system (HS),

$$\Gamma^h := \{(U,V) \in \mathbb{R}^2 : H(U,V) = h\}, \qquad h \in (0,\beta),$$

such that

(i) $\Gamma^h \to P_0 = (0,0)$ as $h \to 0^+$, and

(ii) $\Gamma^h \to \Gamma^\beta$ as $h \to \beta^-$.

If $\widetilde{G}(u) = \int_0^u g(s)\,ds$ then under (H_2) and (H_3) it is easy to check that for each $h \in (0,\beta)$ there exist unique values $u_1(h) \in (u_*, 0)$ and $u_2(h) \in (0,1)$ such that $\widetilde{G}(u_1(h)) = \widetilde{G}(u_2(h)) = h$ and the periodic orbits are given by

$$V(U) = \pm \overline{V}^h(U) := \pm\sqrt{2\Big(h - \int_0^U g(s)\,ds\Big)},$$

for $U \in (u_1(h), u_2(h))$, $h \in (0,\beta)$.

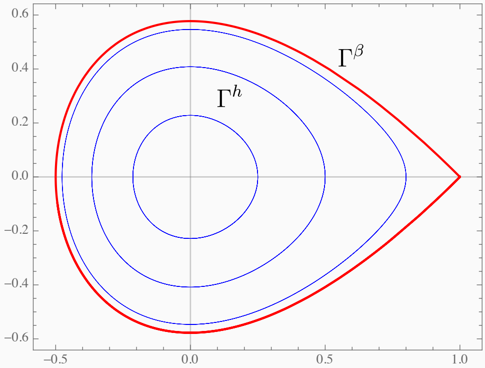

Figure: Homoclinic loop Γ^β in the (U,V)-plane and periodic orbits Γ^h, $h \in (0,\beta)$, for the Hamiltonian system. Here $g(u) = u(1-u)$ (logistic).

Energy levels for the Hamiltonian system (iii)

- If $T = T(h)$ is the fundamental period of the periodic orbit Γ^h, $h \in (0, \beta)$, then $T(h) \to \infty$ as $h \to \beta^-$. The period can be computed explicitly:

$$T(h) = \sqrt{2} \int_{u_1(h)}^{u_2(h)} \frac{dy}{\sqrt{h - \int_0^y g(s)\, ds}}, \qquad h \in (0, \beta).$$

From standard properties of Hamiltonian systems, $0 < T(h) < \infty$ for each $h \in (0, \beta)$ and $T(h) \to \infty$ as $h \to \beta^-$, which is the infinite period of the homoclinic loop Γ^β.

Melnikov integrals for the perturbed system

Define the associated Melnikov integrals for the perturbed system (cf. Han and Yu (2012), Chicone (2006)):

$$\widetilde{M}(h,\mu) := \int_{\mathrm{int}(\Gamma_h)} (\partial_U R + \partial_V Q)\, dU\, dV.$$

They satisfy (see Han and Yu (2012)):

- $\widetilde{M} \in C^\infty$ for $|\varepsilon| + |h - h_0| \ll 1$ small for any $h_0 \in (0,\beta)$, all $\mu \in \mathbb{R}^2$.
- The derivative of $\widetilde{M}$ with respect to h is determined by:

  $$\partial_h \widetilde{M}(h,\mu) = \oint_{\Gamma^h} (\partial_U R + \partial_V Q)\, d\sigma_h, \qquad h \in (0,\beta),\ \mu \in \mathbb{R}^2,$$

  where $d\sigma_h$ denotes the arc length measure on Γ^h.

The Melnikov integrals precisely at $h = \beta$ are

$$M(\mu) := \widetilde{M}(\beta,\mu) = \int_{\mathrm{int}(\Gamma^\beta)} (\partial_U R + \partial_V Q)\, dU\, dV,$$

$$M_1(\mu) := \partial_h \widetilde{M}(\beta,\mu) = \oint_{\Gamma^\beta} (\partial_U R + \partial_V Q)\, d\sigma_\beta.$$

Melnikov's theorem

Theorem (Melnikov's method for perturbed homoclinic orbits)

Suppose that P_1 is a hyperbolic saddle for the unperturbed Hamiltonian system, with a homoclinic loop Γ^{β}. If $\varepsilon > 0$ is small then the perturbed system (PS) *has a unique hyperbolic saddle $P_1(\varepsilon) = P_1 + O(\varepsilon)$. Moreover, if $M(\mu_0) = 0$ and $M_1(\mu_0) \neq 0$ then, for each $\varepsilon > 0$ sufficiently small, the perturbed system with $\mu = \mu_0$ has a unique hyperbolic homoclinic loop $\Gamma_{\varepsilon}^{\beta}$ relative to the stable and unstable manifolds of the hyperbolic saddle $P_1(\varepsilon)$. If $M(\mu)$ has no zeroes and $|\varepsilon| \neq 0$ is small, then the stable and unstable manifolds of $P_1(\varepsilon)$ do not intersect.*

– Classical theorem by Melnikov (1963). See also Wiggins (2003), Chicone (2006).

Existence of a homoclinic loop

Proposition

Under (H_1) - (H_3) and (H_5), system (A) has a unique homoclinic orbit joining the hyperbolic saddle point $P_1 = (1,0)$ with itself for the parameter values $a = 1$ and $c = c_1 = I_1/I_0$.

Proof. Follows upon application of Melnikov's method. Since $R(U,V,\mu) = 0$ and $Q(U,V,\mu) = \mu_1 f'(U)V - \mu_2 V$, we evaluate the Melnikov integrals:

$$\begin{aligned} M(\mu) = \int_{\mathrm{int}(\Gamma^\beta)} (\mu_1 f'(U) - \mu_2)\, dV\, dU &= \int_{u_*}^{1} \int_{-\gamma(U)}^{\gamma(U)} (\mu_1 f'(U) - \mu_2)\, dV\, dU \\ &= 2\Big(\mu_1 \int_{u_*}^{1} f'(U)\gamma(U)\, dU - \mu_2 \int_{u_*}^{1} \gamma(U)\, dU\Big) \\ &= 2\big(\mu_1 I_1 - \mu_2 I_0\big). \end{aligned}$$

Hence, $M(\mu) = 0$ only when

$$\mu_2 = \left(\frac{I_1}{I_0}\right)\mu_1. \qquad (*)$$

Evaluate M_1 at any $\mu \in \mathbb{R}^2$ satisfying (*):

$$
\begin{aligned}
M_1(\mu) = \oint_{\Gamma\beta} (\mu_1 f'(U) - \mu_2)\, d\sigma_\beta &= \mu_1 \oint_{\Gamma\beta} f'(U)\, d\sigma_\beta - \mu_2 \oint_{\Gamma\beta} d\sigma_\beta \\
&= 2\mu_1 \int_{u_*}^{1} f'(U) \sqrt{1+\gamma'(U)^2}\, dU - \mu_2 |\partial\Omega_\beta| \\
&= \mu_1 J - \mu_2 L \\
&= \mu_1 \left(J - \left(\frac{I_1}{I_0}\right) L \right) \neq 0
\end{aligned}
$$

if $\mu_1 \neq 0$ and in view of (H_5). This implies that there is a whole line of simple zeroes of the Melnikov function determined by

$$
\mathscr{C} := \left\{ (\mu_1, \mu_2) \in \mathbb{R}^2 : \mu_2 = \left(\frac{I_1}{I_0}\right) \mu_1 \right\} \setminus \{(0,0)\}.
$$

Now, fix $\varepsilon > 0$ sufficiently small. By Melnikov's theorem, the perturbed system (PS) has a unique hyperbolic point $P_1(\varepsilon) = P_1 + O(\varepsilon)$. But since $P_1 = (1,0)$ is a hyperbolic saddle for system (PS) for any parameter values, we obtain that $P_1(\varepsilon) \equiv P_1 = (1,0)$. Moreover, for any fixed $\mu_0 \in \mathscr{C}$, $\mu_0 \neq (0,0)$, the perturbed system (PS) with $\mu = \mu_0$ has a unique hyperbolic homoclinic loop, Γ_ε^β, relative to the stable and unstable manifolds of the saddle P_1. For this fixed value of $\varepsilon > 0$, let us define

$$\mu_1 := \frac{1}{\varepsilon} > 0, \qquad \mu_2 = \left(\frac{I_1}{I_0}\right)\mu_1,$$

so that $c = \varepsilon\mu_2 = I_1/I_0$ and $a = \varepsilon\mu_1 = 1$. Therefore, $\mu_0 := (\mu_1, \mu_2) = (1/\varepsilon, (I_0/\varepsilon I_1)) \in \mathscr{C} \subset \mathbb{R}^2$ is a bifurcation value for which the Melnikov integral has a simple zero. In this case the critical value for the speed is $c = c_1 := I_1/I_0$. Now, since $M(\mu_0) = 0$ and $M_1(\mu_0) \neq 0$ we conclude that the perturbed system has a unique homoclinic loop Γ_ε^β relative to the stable and unstable manifolds at P_1, for parameter values $a = 1$ and $c = c_1$. $\square$

Corollary (existence of a traveling pulse)

The system (ODE) *has a homoclinic loop for the speed value* $c = c_1 = I_1/I_0$*, which we denote as* $\Gamma_0 := \{(\psi, \psi')(z) : z \in \mathbb{R}\}$*, with* $\psi \in C^3(\mathbb{R})$ *and such that* $(\psi, \psi')(z) \to (1, 0)$ *as* $z \to \pm\infty$*. The convergence is exponential: there exist constants* $C, \kappa > 0$ *such that*

$$|\psi(z) - 1|, |\psi'(z)| \le Ce^{-\kappa|z|}, \qquad \text{as } |z| \to \infty.$$

This homoclinic orbit is associated to a unique (up to translations) traveling pulse solution to (VBL) *of the form* $u(x,t) = \psi(x - c_1 t)$ *and traveling with speed* $c = c_1$*.*

Theorem (Andronov-Leontovich)

Consider a planar system of the form

$$U' = F(U, V, \mu)$$
$$V' = G(U, V, \mu)$$

where F and G are smooth and $\mu \in \mathbb{R}$. Assume that (U_0, V_0) is a hyperbolic saddle for all μ near μ_0; that at $\mu = \mu_0$ the eigenvalues are $\lambda_1(\mu_0) < 0 < \lambda_2(\mu_0)$; and that the system has a homoclinic orbit Γ_0 at the saddle. Let us define the saddle quantity as

$$\Sigma_0 := \lambda_1(\mu_0) + \lambda_2(\mu_0),$$

and suppose that $\Sigma_0 \neq 0$.

Theorem (Andronov-Leontovich (continuation))

Then:

(a) *If* $\Sigma_0 < 0$ *then for sufficiently small* $\mu - \mu_0 > 0$ *there exists a unique stable periodic orbit* $\Gamma(\mu)$ *bifurcating from* Γ_0 *which as* $\mu \to \mu_0^+$ *gets closer to the homoclinic loop at* $\mu = \mu_0$. *When* $\mu < \mu_0$ *there are no periodic orbits.*

(b) *If* $\Sigma_0 > 0$ *then for sufficiently small* $\mu - \mu_0 < 0$ *there exists a unique unstable periodic orbit* $\Gamma(\mu)$ *bifurcating from* Γ_0 *which as* $\mu \to \mu_0^-$ *becomes the homoclinic loop at* $\mu = \mu_0$. *When* $\mu > \mu_0$ *there are no periodic orbit.*

– Classical theorem by Andronov and Leontovich (1937). See, e.g., Shilnokov et al. (2001).

Periodic wavetrains with large fundamental period (i)

Proof of theorem.

Assume (H_1) - (H_3), (H_5) and (H_6). We know that when $c = c_1$, $P_1 = (1, 0)$ is a hyperbolic saddle for system (ODE) and there is a homoclinic orbit joining P_1 with itself. The eigenvalues of the linearization at P_1,

$$\lambda_1(c_1) = \frac{1}{2}(f'(1) - c_1) - \frac{1}{2}\sqrt{(f'(1) - c_1)^2 - 4g'(1)} < 0,$$
$$\lambda_2(c_1) = \frac{1}{2}(f'(1) - c_1) + \frac{1}{2}\sqrt{(f'(1) - c_1)^2 - 4g'(1)} > 0,$$

satisfy $\lambda_1(c_1) < 0 < \lambda_2(c_1)$. Hence the saddle quantity is non-zero, $\Sigma_0 = f'(1) - c_1 \neq 0$, in view of ($H_6$). Andronov-Leontovich's theorem implies there exists $\tilde{\varepsilon}_1 > 0$ small such that, if $f'(1) > c_1$ (respectively, $f'(1) < c_1$) then for each $c \in (c_1 - \tilde{\varepsilon}_1, c_1)$ (respectively, $c \in (c_1, c_1 + \tilde{\varepsilon}_1)$) there exists a unique closed periodic orbit for system (ODE) with fundamental period $T(c)$.

Periodic wavetrains with large fundamental period (ii)

We obtain a family of periodic orbits parametrized by

$$\varepsilon := |c - c_1| \in (0, \tilde{\varepsilon}_1),$$

denoted as $(\overline{U}^{\varepsilon}, \overline{V}^{\varepsilon})(z) =: (\varphi^{\varepsilon}, (\varphi^{\varepsilon})')(z)$, $z \in \mathbb{R}$, with speed value $c(\varepsilon) = c_1 + \varepsilon$ if $f'(1) < c_1$ or $c(\varepsilon) = c_1 - \varepsilon$ if $f'(1) > c_1$, fundamental period T_{ε}, and amplitude $|\varphi^{\varepsilon}(z)|, |(\varphi^{\varepsilon})'(z)| = O(1)$ as $\varepsilon \to 0^+$. Moreover, the family of orbits converge to the homoclinic loop relative to the saddle point $P_1 = (1,0)$ as $\varepsilon \to 0^+$, which we denote as

$$(\varphi^0, (\varphi^0)')(z) := (\psi, \psi')(z), \qquad z \in \mathbb{R},$$

with $(\varphi^0, (\varphi^0)')(z) \to (1,0)$ exponentially fast as $z \to \pm\infty$. Hence, there exists $\widetilde{\delta}(\varepsilon) > 0$ such that $\widetilde{\delta}(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and

$$|\varphi^0(z) - \varphi^{\varepsilon}(z)| \le \widetilde{\delta}(\varepsilon), \qquad \text{for all } |z| \le \frac{T_{\varepsilon}}{2}.$$

Periodic wavetrains with large fundamental period (iii)

It can be shown that the homoclinic loop $(\varphi^0, (\varphi^0)') = (\psi, \psi')$ is non-degenerate (see definition by Beyn (1990)). Apply Corollary 3.2 in Beyn (1990) to conclude that there exists $0 < \varepsilon_1 < \tilde{\varepsilon}_1$ sufficiently small and an appropriate reparametrization of the phase z such that

$$\sup_{z\in[-\frac{T_\varepsilon}{2}, \frac{T_\varepsilon}{2}]} \left(|\varphi^0(z) - \varphi^\varepsilon(z)| + |(\varphi^0)'(z) - (\varphi^\varepsilon)'(z)|\right) \leq C \exp\Big(-\big(\min\{\lambda_2(c_1), |\lambda_1(c_1)|\}\big) \tfrac{T_\varepsilon}{2}\Big),$$

$$\varepsilon \leq C \exp\left(-\big(\min\{\lambda_2(c_1), |\lambda_1(c_1)|\}\big) T_\varepsilon\right),$$

for each $0 < \varepsilon < \varepsilon_1$. Set $\kappa = \min\{\lambda_2(c_1), |\lambda_1(c_1)|\} > 0$. This shows the desired bounds. Notice the second bound implies $T_\varepsilon = O(|\log \varepsilon|) \to \infty$. Finally, the family of orbits φ^ε is of class C^3 in $z \in \mathbb{R}$ and in the bifurcation parameter c, by the regularity of f and g, and to standard ODE results. □

(I) Burgers-Fisher equation

When $g(u) = u(1-u)$ the function $\gamma = \gamma(u)$ is

$$\gamma(u) = \frac{1}{\sqrt{3}}\sqrt{1-3u^2+2u^3}, \qquad u \in (-\tfrac{1}{2}, 1).$$

Hence, I_0 and I_1 can be computed:

$$I_0 = \int_{-\frac{1}{2}}^{1} \gamma(s)\, ds = \frac{1}{\sqrt{3}} \int_{-\frac{1}{2}}^{1} \sqrt{1-3s^2+2s^3}\, ds = \frac{3}{5},$$

$$I_1 = \int_{-\frac{1}{2}}^{1} f'(s)\gamma(s)\, ds = \frac{1}{\sqrt{3}} \int_{-\frac{1}{2}}^{1} s\sqrt{1-3s^2+2s^3}\, ds = \frac{3}{35}.$$

L and J are approximated numerically,

$$L = 2\int_{-\frac{1}{2}}^{1} \sqrt{\frac{1-4s^3+3s^4}{1-3s^2+2s^3}}\, ds \approx 4.0734,$$

$$J = 2\int_{-\frac{1}{2}}^{1} s\sqrt{\frac{1-4s^3+3s^4}{1-3s^2+2s^3}}\, ds \approx 0.6906.$$

The non-degeneracy condition (H_5) holds: $I_0 J \approx 0.4152 \neq L I_1 \approx 0.3492$.

The homoclinic loop speed is

$$c_1 = \frac{I_1}{I_0} = \frac{1}{7}.$$

The saddle condition (H_6) holds: $f'(1) = 1 \neq c_1$.

Since $f'(1) = 1 > c_1 = 1/7$, then the family of periodic waves with large period emerge for $c \in (\frac{1}{7} - \varepsilon_1, \frac{1}{7})$ with $\varepsilon_1 > 0$ small.

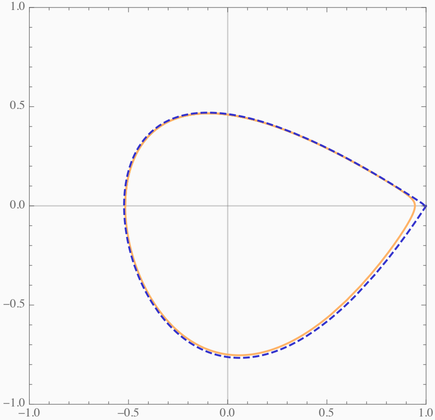

Figure: Numerical approximation of the homoclinic loop for the Burgers-Fisher equation with speed value $c_1 = 1/7$ (in blue, dashed line) and the periodic wave nearby with speed value $c_1 - \varepsilon$, $\varepsilon \approx 0.05$ (solid, orange line).

Examples

(II) Logistic Buckley-Leverett model

Same logistic function, so $I_0 = 3/5$. We also have,

$$I_1 = \int_{-1/2}^{1} f'(s)\gamma(s)\,ds = \int_{-1/2}^{1} \frac{s(1-s)\sqrt{1-3s^2+2s^3}}{\left(s^2+\frac{1}{2}(1-s)^2\right)^2}\,ds = 0.353458.$$

L and J are approximated numerically,

$$L = 2\int_{-1/2}^{1} \sqrt{\frac{1-4s^3+3s^4}{1-3s^2+2s^3}}\;ds \approx 4.0734,$$

$$J = 2\int_{-1/2}^{1} \frac{s(1-s)}{\left(s^2+\frac{1}{2}(1-s)^2\right)^2}\sqrt{\frac{1-4s^3+3s^4}{1-3s^2+2s^3}}\;ds \approx 1.6272.$$

The non-degeneracy condition (H_5) holds: $I_0 J \approx 0.9763 \neq L I_1 \approx 1.4398$.

The homoclinic loop speed is

$$c_1 = \frac{I_1}{I_0} = 0.589097.$$

The saddle condition (H_6) holds: $f'(1) = 0 \neq c_1$.

Since $f'(1) = 0 < c_1$, then the periodic waves emerge for $c \in (0.589097, 0.589097 + \varepsilon_1)$ with $\varepsilon_1 > 0$ small.

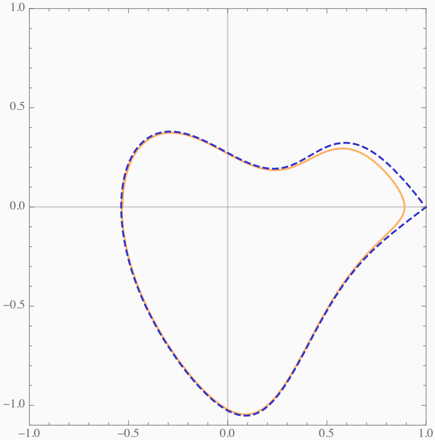

Figure: Numerical approximation of the homoclinic loop for the logistic Buckley-Leverett equation with speed value $c_1 \approx 0.5891$ (in blue, dashed line) and the periodic wave nearby with speed value $c_1 + \varepsilon$, $\varepsilon \approx 0.025$ (solid, orange line).

(III) Modified generalized Burgers-Fisher equation

When $f(u) = \frac{1}{4}u^4 - \frac{1}{3}u^3$ and $g(u) = u - u^4$ we have the approximations

$$I_0 = \frac{1}{\sqrt{5}} \int_{u_*}^{1} \sqrt{3 - 5s^2 + 2s^5}\, ds \approx 0.979027,$$
$$I_1 = \frac{1}{\sqrt{5}} \int_{u_*}^{1} (s^3 - s^2)\sqrt{3 - 5s^2 + 2u^5}\, ds \approx -0.129571,$$

and,

$$L = 2\int_{u_*}^{1} \sqrt{\frac{3 - 8s^5 + 5s^8}{3 - 5s^2 + 2s^5}}\, ds \approx 5.02904,$$
$$J = 2\int_{u_*}^{1} (s^3 - s^2)\sqrt{\frac{3 - 8s^5 + 5s^8}{3 - 5s^2 + 2s^5}}\, ds \approx -1.27529,$$

The non-degeneracy condition (H_5) holds:
$I_0 J \approx -1.24854 \neq L I_1 \approx -0.65162$.

The homoclinic loop speed is

$$c_1 = \frac{I_1}{I_0} \approx -0.13235,$$

The saddle condition (H_6) holds: $f'(1) = 0 \neq c_1$.

Since $f'(1) = 0 > c_1 = -0.13235$, then the periodic waves emerge for $c \in (-0.13235 - \varepsilon_1, -0.13235)$ with $\varepsilon_1 > 0$ small.

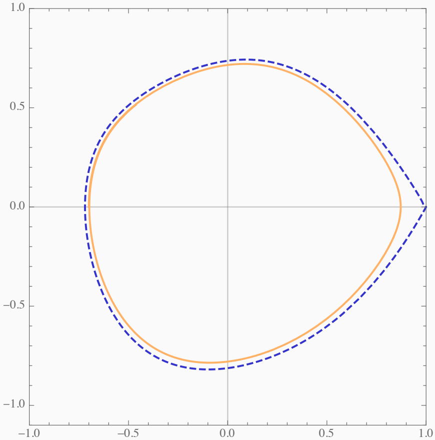

Figure: Numerical approximation of the homoclinic loop for the modified Burgers-Fisher equation with speed value $c_1 \approx -0.13235$ (in blue, dashed line) and the periodic wave nearby with speed value $c_1 - \varepsilon$, $\varepsilon \approx 0.05$ (solid, orange line).

Spectral instability

The spectral stability problem

Spectral stability refers to an important property of the traveling wave as a solution to the PDE.

Consider a perturbation v of a bounded periodic traveling wave $\varphi = \varphi(z)$, $z = x - ct$, with speed c and fundamental period $T > 0$. Substituting $v + \varphi$ into the viscous balance law (VBL) written in the variables $(z, t) = (x - ct, t)$, then $v = v(z, t)$ is a solution to

$$v_t - cv_z + f(v + \varphi)_z - f(\varphi)_z = v_{zz} + g(v + \varphi) - g(\varphi).$$

For nearby perturbations the leading approximation is given by the linearization around φ:

$$v_t = v_{zz} + (c - f'(\varphi))v_z + (g'(\varphi) - f'(\varphi)_z)v.$$

Take $v(z,t) = e^{\lambda t}w(z)$, where $\lambda \in \mathbb{C}$ (growth rate) and $w \in X$ Banach, we obtain an eigenvalue problem:

$$\lambda w = w_{zz} + (c - f'(\varphi))w_z + (g'(\varphi) - f'(\varphi)_z)w.$$

Our choice: $X = L^2(\mathbb{R};\mathbb{C})$, stability with respect to small localized perturbations.

The linearized operator around the wave is

$$\begin{cases} \mathscr{L} : L^2(\mathbb{R};\mathbb{C}) \longrightarrow L^2(\mathbb{R};\mathbb{C}), \\ \mathscr{L} := \partial_z^2 + a_1(z)\partial_z + a_0(z)\mathbb{I}, \end{cases} \tag{LO}$$

with dense domain $\mathscr{D}(\mathscr{L}) = H^2(\mathbb{R};\mathbb{C})$, and where the coefficients,

$$\begin{aligned} a_1(z) &:= c - f'(\varphi), \\ a_0(z) &:= g'(\varphi) - f'(\varphi)_z, \end{aligned}$$

are bounded and periodic, $a_j(z+T) = a_j(z)$, $\forall\, z \in \mathbb{R}$, $j = 0, 1$.

Definition (resolvent and spectra)

Let $\mathscr{L} : X \to Y$ be a closed linear operator, with X, Y Banach and dense domain $\mathscr{D}(\mathscr{L}) \subset X$. The resolvent of $\mathscr{L}$, $\rho(\mathscr{L})$, is the set of all complex numbers $\lambda \in \mathbb{C}$ such that $\mathscr{L} - \lambda$ is injective and onto, and $(\mathscr{L} - \lambda)^{-1}$ is bounded. The point spectrum of $\mathscr{L}$, $\sigma_{\mathrm{pt}}(\mathscr{L})$, is the set of $\lambda \in \mathbb{C}$ such that $\mathscr{L} - \lambda$ is a Fredholm operator with index zero and non-trivial kernel. The essential spectrum of $\mathscr{L}$, $\sigma_{\mathrm{ess}}(\mathscr{L})$, is the set of all $\lambda \in \mathbb{C}$ such that either $\mathscr{L} - \lambda$ is not Fredholm, or it is Fredholm with non-zero index. The spectrum of $\mathscr{L}$ is defined as $\sigma(\mathscr{L}) = \sigma_{\mathrm{ess}}(\mathscr{L}) \cup \sigma_{\mathrm{pt}}(\mathscr{L})$.

Definition (spectral stability)

We say that a bounded periodic wave φ is spectrally stable as a solution to the viscous balance law (VBL) if the L^2-spectrum of the linearized operator around the wave defined in (LO) satisfies

$$\sigma(\mathscr{L})_{|L^2} \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} = \varnothing.$$

Otherwise we say that it is spectrally unstable.

Floquet characterization of the spectrum

Well-known fact: since the coefficients $\mathscr{L}$ are periodic in z, Floquet theory implies that the L^2-spectrum is purely essential or "continuous", $\sigma(\mathscr{L})_{|L^2} = \sigma_{\mathrm{ess}}(\mathscr{L})_{|L^2}$, and there are no isolated eigenvalues.

We can parametrize the spectrum in terms of Floquet multipliers of the form $e^{i\theta} \in \mathbb{S}^1$ (or by $\theta \in \mathbb{R} \pmod{2\pi}$). Define the set σ_θ as the set of complex numbers λ for which there exists a bounded, non-trivial solution $w \in L^\infty(\mathbb{R};\mathbb{C})$ to the boundary value problem

$$\begin{cases} \lambda w = w_{zz} + \big(c - f'(\varphi)\big) w_z + \big(g'(\varphi) - f'(\varphi)_z\big) w, \\ w(T) = e^{i\theta} w(0), \\ w_z(T) = e^{i\theta} w_z(0), \end{cases}$$

for some $\theta \in (-\pi, \pi]$.

Floquet characterization of the spectrum (ii)

Define the Floquet spectrum σ_F as:

$$\sigma_F := \bigcup_{-\pi<\theta\leq\pi} \sigma_\theta.$$

Lemma (Floquet characterization of the spectrum)

$\sigma(\mathscr{L})_{|L^2} = \sigma_F.$

See Jones et al. (2014); Gardner (1993).

Observation: The continuous spectrum $\sigma(\mathscr{L})_{|L^2}$ can be written as the union of partial spectra σ_θ. Each set σ_θ is discrete as it is the zero set of an analytic function. If $\theta = 0$ then the boundary conditions become periodic and σ_0 detects perturbations which are co-periodic. The set σ_π detects anti-periodic perturbations.

Bloch-wave decomposition

Define $u(z) := e^{-i\theta z/T} w(z)$. Then the non-separated boundary conditions transform into periodic ones, $\partial_z^j u(T) = \partial_z^j u(0)$, $j = 0, 1$, and the spectral problem is recast as $\mathscr{L}_\theta u = \lambda u$ for a one-parameter family of Bloch operators:

$$\begin{cases} \mathscr{L}_\theta := (\partial_z + i\theta/T)^2 + a_1(z)(\partial_z + i\theta/T) + a_0(z)\mathbb{I}, \\ \mathscr{L}_\theta : L^2_{\mathrm{per}}([0,T];\mathbb{C}) \to L^2_{\mathrm{per}}([0,T];\mathbb{C}), \end{cases}$$

with domain $\mathscr{D}(\mathscr{L}_\theta) = H^2_{\mathrm{per}}([0,T];\mathbb{C})$, parametrized by $\theta \in (-\pi, \pi]$.

Their spectrum consists entirely of isolated eigenvalues: $\sigma(\mathscr{L}_\theta)_{|L^2_{\mathrm{per}}} = \sigma_{\mathrm{pt}}(\mathscr{L}_\theta)_{|L^2_{\mathrm{per}}}$. Moreover, they depend continuously on the Bloch parameter θ, which is a local coordinate for the spectrum $\sigma(\mathscr{L})_{|L^2}$

Conclusion: $\lambda \in \sigma(\mathscr{L})_{|L^2}$ if and only if $\lambda \in \sigma_{\mathrm{pt}}(\mathscr{L}_\theta)_{L^2_{\mathrm{per}}}$ for some $\theta \in (-\pi, \pi]$:

$$\sigma(\mathscr{L})_{|L^2} = \bigcup_{-\pi < \theta \le \pi} \sigma_{\mathrm{pt}}(\mathscr{L}_\theta)_{|L^2_{\mathrm{per}}}.$$

Spectral instability of small-amplitude waves

Family of periodic, small-amplitude waves parametrized by $\varepsilon := |c - c_0| \in (0, \varepsilon_0)$, $c_0 = f'(0)$. These waves have amplitude of order $|\varphi^\varepsilon|, |\varphi_z^\varepsilon| = O(\sqrt{\varepsilon})$ and period $T_\varepsilon = \frac{2\pi}{\sqrt{g'(0)}} + O(\varepsilon) =: T_0 + O(\varepsilon)$.

The associated spectral problem can be recast in a periodic space. Consider the following Bloch-type transformation,

$$y := \frac{\pi z}{T_\varepsilon}, \quad u(y) := e^{-i\theta y/\pi} w\Big(\frac{T_\varepsilon y}{\pi}\Big),$$

for given $\theta \in (-\pi, \pi]$. Then the spectral problem is now

$$\lambda u = \frac{1}{T_\varepsilon^2}\big(i\theta + \pi\partial_y\big)^2 u + \frac{\bar{a}_1^\varepsilon(y)}{T_\varepsilon}\big(i\theta + \pi\partial_y\big)u + \bar{a}_1^\varepsilon(y)u,$$

where the coefficients

$$\bar{a}_1^\varepsilon(y) := c(\varepsilon) - f'(\varphi^\varepsilon(T_\varepsilon y/\pi)),$$
$$\bar{a}_0^\varepsilon(y) := g'(\varphi^\varepsilon(T_\varepsilon y/\pi)) - f''(\varphi^\varepsilon(T_\varepsilon y/\pi))\varphi_z^\varepsilon(T_\varepsilon y/\pi),$$

are clearly π-periodic in the y variable and where $u \in H^2_{\rm per}([0,\pi];\mathbb{C})$ is subject to π-periodic boundary conditions, $u(0) = u(\pi)$, $u_y(0) = u_y(\pi)$.

Multiply by T_ε^2 (constant) to obtain the following equivalent spectral problem

$$\mathscr{L}_\theta u = \widetilde{\lambda}\, u, \tag{nL}$$

for the operator

$$\begin{cases} \mathscr{L}_\theta := \big(i\theta + \pi\partial_y\big)^2 + a_1^\varepsilon(y)\big(i\theta + \pi\partial_y\big) + a_0^\varepsilon(y)\mathbb{I}, \\ \mathscr{L}_\theta : \mathscr{D}(\mathscr{L}_\theta) = H^2_{\text{per}}([0,\pi];\mathbb{C}) \subset L^2_{\text{per}}([0,\pi];\mathbb{C}) \longrightarrow L^2_{\text{per}}([0,\pi];\mathbb{C}), \end{cases}$$

for any given $\theta \in (-\pi,\pi]$ and where

$$\widetilde{\lambda} := T_\varepsilon^2\, \lambda,$$

$$a_1^\varepsilon(y) := T_\varepsilon\, \bar{a}_1^\varepsilon(y),$$

$$a_0^\varepsilon(y) := T_\varepsilon^2\, \bar{a}_0^\varepsilon(y).$$

Perturbation problem

Let us write (nL) as a perturbation problem. The coefficients can be written as

$$
\begin{aligned}
a_1^\varepsilon(y) &= \big(T_0 + O(\varepsilon)\big)\big(c(\varepsilon) - f'(\varphi^\varepsilon(T_\varepsilon y/\pi))\big) = \sqrt{\varepsilon}\, b_1(y) \\
a_0^\varepsilon(y) &= \big(T_0 + O(\varepsilon)\big)^2 \Big(g'(\varphi^\varepsilon(T_\varepsilon y/\pi)) - f''(\varphi^\varepsilon(T_\varepsilon y/\pi))\varphi_z^\varepsilon(T_\varepsilon y/\pi)\Big) \\
&= 4\pi^2 + O(\sqrt{\varepsilon}).
\end{aligned}
$$

where

$$
b_1(y) := \frac{1}{\sqrt{\varepsilon}}\, a_1^\varepsilon(y) = O(1), \quad y \in [0, \pi].
$$

Thus we write

$$
b_0(y) := \frac{a_0^\varepsilon(y) - 4\pi^2}{\sqrt{\varepsilon}} = O(1), \quad y \in [0, \pi].
$$

Now, if we denote $\eta := \sqrt{\varepsilon} \in (0, \sqrt{\varepsilon_0})$ we obtain

$$\begin{aligned}\mathscr{L}_\theta u &= \left(i\theta + \pi\partial_y\right)^2 u + 4\pi^2 u + \eta\, b_1(y)\left(i\theta + \pi\partial_y\right)u + \eta\, b_0(y)u\\ &= \mathscr{L}_\theta^0 u + \eta \mathscr{L}_\theta^1 u,\end{aligned}$$

where the operators $\mathscr{L}_\theta^0$ and $\mathscr{L}_\theta^1$ are defined as

$$\begin{cases}\mathscr{L}_\theta^0 := \left(i\theta + \pi\partial_y\right)^2 + 4\pi^2\mathbb{I},\\ \mathscr{L}_\theta^0 : \mathscr{D}(\mathscr{L}_\theta^0) = H^2_{\mathrm{per}}([0,\pi],\mathbb{C}) \subset L^2_{\mathrm{per}}([0,\pi],\mathbb{C}) \longrightarrow L^2_{\mathrm{per}}([0,\pi],\mathbb{C}),\end{cases}$$

$$\begin{cases}\mathscr{L}_\theta^1 := b_1(y)\left(i\theta + \pi\partial_y\right) + b_0(y)\mathbb{I},\\ \mathscr{L}_\theta^1 : \mathscr{D}(\mathscr{L}_\theta^1) = H^1_{\mathrm{per}}([0,\pi],\mathbb{C}) \subset L^2_{\mathrm{per}}([0,\pi],\mathbb{C}) \longrightarrow L^2_{\mathrm{per}}([0,\pi],\mathbb{C}).\end{cases}$$

Therefore, (nL) is recast as a perturbed spectral problem of the form

$$\mathscr{L}_\theta u = \mathscr{L}_\theta^0 u + \eta \mathscr{L}_\theta^1 u = \widetilde{\lambda}\, u, \qquad u \in H^2_{\mathrm{per}}([0,\pi],\mathbb{C}).$$

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$$\mathscr{L}_\theta u = \mathscr{L}_\theta^0 u + \eta \mathscr{L}_\theta^1 u = \widetilde{\lambda} u, \qquad u \in H^2_{\mathrm{per}}([0,\pi],\mathbb{C}).$$

Lemma

For each $\theta \in (-\pi,\pi]$, $\mathscr{L}_\theta^1$ is $\mathscr{L}_\theta^0$-bounded.

Proof. We need to show that there exist uniform constants $\alpha, \beta \geq 0$ such that

$$\|\mathscr{L}_\theta^1 u\|_{L^2_{\mathrm{per}}} \leq \alpha \|u\|_{L^2_{\mathrm{per}}} + \beta \|\mathscr{L}_\theta^0 u\|_{L^2_{\mathrm{per}}},$$

for all $u \in H^2_{\mathrm{per}}([0,\pi];\mathbb{C})$.

Upon estimation,

$$\begin{aligned}\|\mathscr{L}_\theta^1 u\|_{L^2_{\mathrm{per}}} &= \|b_1(y)\big(i\theta+\partial_y\big)u + b_0(y)u\|_{L^2_{\mathrm{per}}}\\ &\leq \pi K_1\|u_y\|_{L^2_{\mathrm{per}}} + (\pi K_1 + K_0)\|u\|_{L^2_{\mathrm{per}}},\end{aligned}$$

as $|\theta|\leq\pi$, with $0<K_1:=\|b_1\|_{L^\infty}$, $0<K_0:=\|b_0\|_{L^\infty}$.

It can be shown that for all $u\in H^2([0,\pi];\mathbb{C})$,

$$\|u_y\|_{L^2(0,\pi)} \leq \frac{\pi}{N-1}\|u_{yy}\|_{L^2(0,\pi)} + \frac{2N(N+1)}{\pi(N-1)}\|u\|_{L^2(0,\pi)},$$

with N any positive number with $N>1$.

Upon substitution,

$$\|\mathscr{L}_\theta^1 u\|_{L^2_{\mathrm{per}}} \leq C_1(N)\|u_{yy}\|_{L^2_{\mathrm{per}}} + C_0(N)\|u\|_{L^2_{\mathrm{per}}},$$

where

$$\begin{aligned}C_1(N) &= \frac{\pi^2 K_1}{N-1} > 0,\\ C_0(N) &= K_0 + \frac{K_1}{N-1}\Big(\pi(N-1)+2N(N+1)\Big) > 0.\end{aligned}$$

Using

$$\|\mathscr{L}_\theta^0 u\|_{L^2_{\text{per}}} = \|(i\theta+\pi\partial_y)^2 u + 4\pi^2 u\|_{L^2_{\text{per}}} \geq \pi^2 \|u_{yy}\|_{L^2_{\text{per}}} - \|2i\theta\pi u_y + (4\pi^2-\theta^2)u\|_{L^2_{\text{per}}},$$

and choosing N sufficiently large, $N \geq 1+4\pi$, we arrive at

$$\|\mathscr{L}_\theta^1 u\|_{L^2_{\text{per}}} \leq \alpha \|u\|_{L^2_{\text{per}}} + \beta \|\mathscr{L}_\theta^0 u\|_{L^2_{\text{per}}},$$

with uniform constants

$$\alpha := \frac{8C_1(N)}{\pi(N-1)}\Big(\pi(N-1) + N(N+1)\Big) + C_0(N) > 0,$$
$$\beta := \frac{2C_1(N)}{\pi^2} > 0.$$

□

Analytic perturbation theory (i)

Take a look at the spectral problem specialized to the case of Bloch parameter $\theta = 0$,

$$\mathscr{L}_0 u = \mathscr{L}_0^0 + \eta \mathscr{L}_0^1 u = \widetilde{\lambda}\, u, \qquad u \in H^2_{\mathrm{per}}([0,\pi];\mathbb{C}).$$

Observe:

- The operator

$$\begin{cases} \mathscr{L}_0^0 = \pi^2 \partial_y^2 + 4\pi^2 \mathbb{I}, \\ \mathscr{L}_0^0 : L^2_{\mathrm{per}}([0,\pi];\mathbb{C}) \to L^2_{\mathrm{per}}([0,\pi];\mathbb{C}), \end{cases}$$

  is self-adjoint with a positive eigenvalue $\widetilde{\lambda}_0 = 4\pi^2$ associated to the constant eigenfunction $u_0(y) = 1/\sqrt{\pi}$.

Analytic perturbation theory (i)

- Upon application of analytic perturbation theory of linear operators (Kato, 1980; Hislop and Segal, 1996): $\mathscr{L}_0^1$ is $\mathscr{L}_0^0$-bounded implies that $\mathscr{L}_0 = \mathscr{L}_0^0 + \eta\mathscr{L}_0^1$ has discrete eigenvalues $\widetilde{\lambda}_j(\eta)$ in a η-neighborhood of $\widetilde{\lambda}_0 = 4\pi^2$ with multiplicities adding up to m_0 if η is sufficiently small.
- Moreover, since $\widetilde{\lambda}_0 > 0$ there holds

$$\operatorname{Re}\lambda_j(\eta) > 0, \qquad |\eta| \ll 1.$$

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- Moreover, since $\widetilde{\lambda}_0 > 0$ there holds

$$\operatorname{Re} \lambda_j(\eta) > 0, \qquad |\eta| \ll 1.$$

Lemma

For each $0 < \eta \ll 1$ sufficiently small there holds

$$\sigma_{\mathrm{pt}}(\mathscr{L}_0^0 + \eta \mathscr{L}_0^1)_{|L^2_{\mathrm{per}}} \cap \{\lambda \in \mathbb{C} : |\lambda - 4\pi^2| < r(\eta)\} \neq \varnothing,$$

for some $r(\eta) = O(\eta) > 0$.

Proof of theorem. Now, since $\eta = \sqrt{\varepsilon}$, we conclude that for $0 < \varepsilon \ll 1$ sufficiently small there exist discrete eigenvalues $\lambda(\varepsilon) \in \sigma_{\mathrm{pt}}(\mathscr{L}_0^0 + \sqrt{\varepsilon}\mathscr{L}_0^1)$ such that $|\lambda - 4\pi^2| \leq C\sqrt{\varepsilon}$ for some $C > 0$. Transforming back into the original problem, this implies that there exist eigenvalues $\lambda = \lambda(\varepsilon)$ that satisfy

$$|\lambda(\varepsilon) - g'(0)| = O(\sqrt{\varepsilon}), \qquad 0 < \varepsilon \ll 1.$$

This implies that for $\varepsilon > 0$ small there exist unstable eigenvalues $\lambda(\varepsilon)$ with $\mathrm{Re}\,\lambda(\varepsilon) > 0$ of the spectral problem with $\theta = 0$. Let θ vary within $(-\pi, \pi]$ to obtain curves of spectrum that locally remain in the unstable half plane. We conclude that

$$\sigma(\mathscr{L}^\varepsilon)_{|L^2} = \bigcup_{-\pi<\theta\leq\pi} \sigma(\mathscr{L}_\theta)_{|L^2_{\mathrm{per}}} \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\} \neq \varnothing,$$

for $\varepsilon > 0$ sufficiently small. $\square$

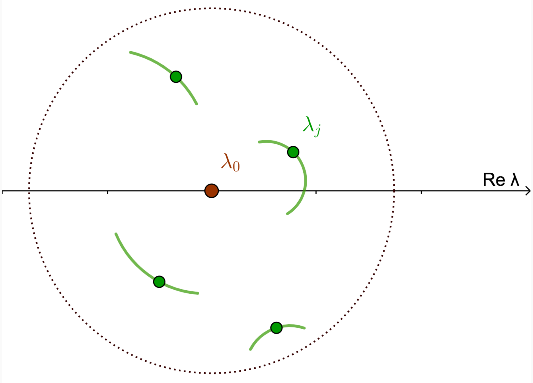

Figure: Cartoon representation of the unstable real eigenvalue $\lambda_0 = g'(0) > 0$ (in red) and of the neighboring unstable eigenvalues $\lambda_j(\varepsilon)$ (in green) near λ_0 for $0 < \varepsilon \ll 1$ small for the case of a Floquet exponent $\theta = 0$. By letting θ vary within $(-\pi, \pi]$ we obtain unstable curves of spectrum (in green) of the linearized operator around the periodic wave.

Observation:

The “instablility” of $u = 0$ as equilibrium point of the reaction function ($g'(0) > 0$) is responsible for the spectral instability of the small amplitude waves bifurcating from the equilibrium. Heuristically, this result can be interpreted as follows: when $\varepsilon \to 0^+$ the small-amplitude periodic waves collapse to the origin and the linearized operator tends (formally) to a constant coefficient linearized operator around zero, whose spectrum is determined by a dispersion relation that invades the unstable half plane thanks to the sign of $g'(0)$.

Spectral instability of large period waves

Consider the family of large period waves,

$$u(x,t) = \varphi^\varepsilon(x - c(\varepsilon)t),$$
$$\varphi^\varepsilon(z) = \varphi^\varepsilon(z + T_\varepsilon), \qquad \text{for all } \; z \in \mathbb{R},$$

traveling with speed $c = c(\varepsilon)$ and parametrized by $\varepsilon = |c_1 - c(\varepsilon)|$, with $0 < \varepsilon < \varepsilon_1 \ll 1$.

The family converges as $\varepsilon \to 0^+$ to the traveling pulse $\varphi^0 = \varphi^0(x - c_1 t)$ traveling with speed $c_1 = I_1/I_0$ (homoclinic orbit). The fundamental period of the family of periodic waves, $T_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ at order $O(|\log \varepsilon|)$.

The homoclinic Evans function (i)

Suppose $\varphi = \varphi^0(z)$, $z = x - c_1 t$ is the homoclinic pulse, with $u_\pm = \lim_{z\to\pm\infty} \varphi(z) = 1$. Then the eigenvalue problem $\bar{\mathscr{L}}^0 u = \lambda u$ can be recast as

$$W_z = \mathbf{A}^0(z,\lambda) W, \tag{SS}$$

with

$$W = \begin{pmatrix} u \\ u_z \end{pmatrix}, \qquad \mathbf{A}^0(z,\lambda) := \begin{pmatrix} 0 & 1 \\ \lambda - (g'(\varphi^0) - f'(\varphi^0)_z) & -c_1 + f'(\varphi^0) \end{pmatrix}.$$

Then, for each λ in the set of consistent splitting,

$$\Omega_\infty = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > g'(1)\},$$

there exists one solution $W^+(z,\lambda)$ spanning the stable space of (SS) decaying as $z \to +\infty$, and one unstable solution $W^-(z,\lambda)$ decaying as $z \to -\infty$.

The homoclinic Evans function (ii)

The The homoclinic Evans function is defined as the Wronskian

$$D(\lambda) := \det(W^{-}(z,\lambda),\, W^{+}(z,\lambda))_{|z=0}.$$

Properties:

- it is not unique but they all differ by appropriate non-vanishing factors;
- D is analytic on Ω_∞;
- it vanishes at $\lambda \in \Omega_\infty$ if and only if $\lambda \in \sigma_{\mathrm{pt}}(\mathscr{L})_{|L^2}$, with the order of the zero being the algebraic multiplicity of the eigenvalue.

– See Kapitula and Promislow (2013), Sandstede (2002).

The periodic Evans function (i)

In the case of a periodic wave $\varphi = \varphi(z)$, with period T, the matrix $\mathbf{A}(z,\lambda)$ is T-periodic in z and we may apply Floquet theory for ODEs.

Let $\mathbf{F} = \mathbf{F}(z,\lambda)$ denote fundamental solution with initial condition $\mathbf{F}(0,\lambda) = \mathbb{I}$ for every $\lambda \in \mathbb{C}$. The T-periodicity in z of the coefficients $\mathbf{A}$ then implies that $\mathbf{F}(z+T,\lambda) = \mathbf{F}(z,\lambda)\mathbf{M}(\lambda)$ for all $z \in \mathbb{R}$, where $\mathbf{M}(\lambda) := \mathbf{F}(T,\lambda)$ is the monodromy matrix, and it is an entire function of $\lambda \in \mathbb{C}$.

It can be shown that $\lambda \in \sigma(\mathscr{L})_{|L^2}$ if and only if there exists $\mu \in \mathbb{C}$ with $|\mu| = 1$ such that

$$\det(\mathbf{M}(\lambda) - \mu\mathbb{I}) = 0.$$

(At least one of the eigenvalues of the monodromy matrix, also known as Floquet multipliers, lies in complex unit circle.)

The periodic Evans function (ii)

Gardner (1993) defines the periodic Evans function as the restriction of the above determinant to μ in the unit circle $\mathbb{S}^1 \subset \mathbb{C}$,

$$D(\lambda, \theta) := \det(\mathbf{M}(\lambda) - e^{i\theta}\mathbb{I}).$$

For each $\theta \in \mathbb{R} \pmod{2\pi}$, the periodic Evans function is an entire function of $\lambda \in \mathbb{C}$ whose isolated zeroes are particular points of the (continuous) spectrum $\lambda \in \sigma(\mathscr{L})_{|L^2}$. Each $\theta \in (-\pi, \pi]$ is precisely the Bloch parameter associated to a Floquet multiplier of the form $e^{i\theta}$. For each θ fixed, the zeroes of the analytic function $D(\lambda, \theta)$ are discrete and coincide in order (multiplicity) and location with the discrete Bloch spectrum, $\sigma_{\mathrm{pt}}(\mathscr{L}_\theta)_{|L^2_{\mathrm{per}}}$.

– See Kapitula and Promislow (2013), Gardner (1993).

Instability of the pulse

Theorem

The traveling pulse solution is spectrally unstable: there exists $\bar{\lambda}_0 > 0$ *(real and strictly positive) such that* $\bar{\lambda}_0 \in \sigma_{\mathrm{pt}}(\bar{\mathscr{L}}^0)$*. Moreover, this eigenvalue is simple.*

Instability of the pulse

Theorem

The traveling pulse solution is spectrally unstable: there exists $\bar{\lambda}_0 > 0$ *(real and strictly positive) such that* $\bar{\lambda}_0 \in \sigma_{\mathrm{pt}}(\bar{\mathscr{L}}^0)$. *Moreover, this eigenvalue is simple.*

This is a well-known fact from Sturm-Liouville theory (see Kapitula and Promislow (2013)).

Approximation theorem for large spatial period (i)

Upon homoclinic bifurcation and approximation as $\varepsilon \to 0^+$ of the large period waves, one can show that for every $|\lambda| \leq M$ (constant), there holds

$$
\begin{aligned}
T_\varepsilon = O(|\log \varepsilon|) \to \infty, &\quad \text{as } \varepsilon \to 0^+, \\
|\mathbf{A}^0(z,\lambda) - \mathbf{A}^0_\infty| \leq C(M) e^{-\bar{\theta}|z|}, &\quad \text{for all } z \in \mathbb{R}, \\
|\mathbf{A}^0(z,\lambda) - \mathbf{A}^\varepsilon(z,\lambda)| \leq C(M) e^{-\kappa T_\varepsilon/2}, &\quad \text{for all } |z| \leq \frac{T_\varepsilon}{2},
\end{aligned}
\tag{**}
$$

for some uniform constants $C(M), \kappa, \bar{\theta} > 0$.

Approximation theorem for large spatial period (Ii)

Conditions (**) are the structural assumptions for convergence of periodic spectra in the infinite-period limit to that of the underlying homoclinic wave:

Theorem (Gardner (1997); Yang and Zumbrun (2019))

Assume (**). Then on a compact set $K \subset \Omega_\infty$ *such that the homoclinic Evans function* $D^0 = D^0(\lambda)$ *does not vanish on* ∂K*, the spectra of* $\mathscr{L}^\varepsilon$ *for* T_ε *sufficiently large (or equivalently, for any* $0 < \varepsilon < \varepsilon_2$ *with* $0 < \varepsilon_2 \ll 1$ *sufficiently small) consists of loops of spectra* $\Lambda^\varepsilon_{k,j} \subset \mathbb{C}$*,* $k = 1, \dots, m_j$*, in a neighborhood of order* $O(e^{-\eta T_\varepsilon/(2m_j)})$ *of the eigenvalues* λ_j *of* $\bar{\mathscr{L}}^0$*, where* m_j *denotes the algebraic multiplicity of* λ_j *and* $0 < \eta < \min\{\kappa, \bar{\theta}\}$*.*

Spectral instability of large period waves

Proof of theorem.

Under (H_1) - (H_6), the family of periodic waves with large period, φ^{ε}, as well as the traveling pulse, φ^0, satisfy (**).

Let $\bar{\lambda}_0 > 0$ be the real, simple and positive eigenvalue associated to the traveling pulse. Since $\mathbb{C}_+ \subset \Omega_\infty$ and $\bar{\lambda}_0 > 0$ is simple, then we can take a closed contour Γ around $\bar{\lambda}_0$ such that $K = \bar{\Gamma} \cup (\mathrm{int}\,\Gamma)$ is a small compact set contained in Ω_∞ with no eigenvalues of $\bar{\mathscr{L}}^0$ on $\partial K = \Gamma$. Then from the approximation theorem: there exists $\bar{\varepsilon}_1 := \min\{\varepsilon_1, \varepsilon_2\} > 0$ small such that for all $0 < \varepsilon < \bar{\varepsilon}_1$ there exists a loop of Floquet spectrum $\Lambda^{\varepsilon} \subset \mathbb{C}$ in a small neighborhood around $\bar{\lambda}_0$ of order $O(e^{-\kappa T_\varepsilon/2}) = O(\varepsilon)$ of eigenvalues of the linearized operator $\mathscr{L}^{\varepsilon}$ around φ^{ε}.

This loop does not necessarily contain $\bar{\lambda}_0$ but belongs to a $O(\varepsilon)$-neighborhood. We conclude that the spectrum $\mathscr{L}^{\varepsilon}$ for each periodic wave φ^{ε} with $0 < \varepsilon < \bar{\varepsilon}_1$ is contained in the unstable half plane. $\square$

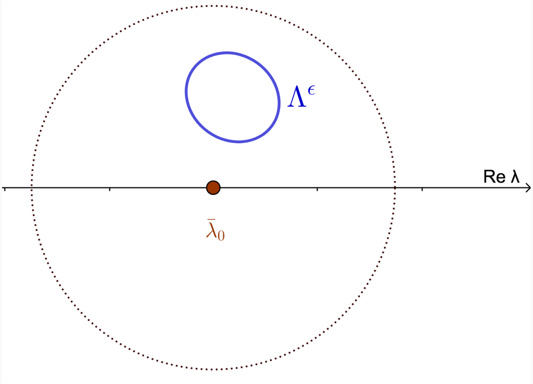

Figure: Cartoon representation of the unstable, simple, real eigenvalue, $\bar{\lambda}_0 > 0$ (in red), of the linearized operator $\bar{\mathscr{L}}^0$ around the homoclinic loop. For $0 < \varepsilon \ll 1$ sufficiently small there exists a unique loop of spectra, Λ^ε (in blue), of the linearized operator $\mathscr{L}^\varepsilon$ around the periodic wave inside an unstable $O(\varepsilon)$-neighborhood of $\bar{\lambda}_0$.

Conclusions

Conclusions

- With classical bifurcation techniques it is possible to deduce the existence of periodic bounded waves for a large class of equations (viscous balance laws).
- One family of small amplitude waves (Hopf bifurcation).
- One family of large period waves (homoclinic bifurcation).
- We also examine the Floquet spectrum of the linearization around both families.
- By analytic perturbation theory (small-amplitude waves) or homoclinic approximation theory (large period waves) we found that both families are spectrally unstable.
- For many equations with structure (Hamiltonian, completely integrable), spectral stability is a prerequisite for their nonlinear (orbital) instability.
- The same waves are modulationally **stable** for the Burgers-Fisher equation.

Reference

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Obrigado!