Dissipative structure of nonlinear viscous dispersive systems

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Motivation

Decay structure for nonlinear higher order systems

- We consider regularizations of first order conservation laws.
- These consist of relaxation, viscous, dispersive, or higher order terms.
- Motivated by parabolic systems, **Kawashima (1983)** found sufficient and necessary conditions for the dissipative structure of such regularizations, yielding a decay structure for equilibrium solutions.
- Applications include:
  - fluid dynamics (**Kawashima, Shizuta, 1985, 1988**)
  - reaction-diffusion systems with transport terms (**Chae, 2018**)
  - thermal relaxation (**Angeles, Malaga, P, 2020)**)
  - fluids with capillarity (**Valdovinos, P, 2022; Kawashima, et al., 2022**), etc.

Consequences:

- Global decay of perturbations to equilibrium states (**Kawashima, Shizuta (1988)**; **P, Valdovinos (2022)**)
- Stability of small amplitude shock profiles:
    - viscous (**Humpherys, Zumbrun, 2002**)
    - relaxation (**Mascia, Zumbrun, 2006**)
    - elliptic coupling (**Nguyen, P, Zumbrun 2010**)
- Historically the first application in mind was the compressible Navier-Stokes-Fourier system.

Compressible Navier-Stokes equations in 1d

Compressible Navier-Stokes-Fourier system (cNSF) in one space dimension:

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x\big(\rho u^2 + \bar{p}\big) = \partial_x\big((\mu + 2\lambda) u_x\big)$$
$$\partial_t\big(\rho(e + \tfrac{1}{2}u^2)\big) + \partial_x\big(\rho u(e + \tfrac{1}{2}u^2) + \bar{p}u\big) = \partial_x\big((\mu + 2\lambda) u u_x\big) + \kappa \theta_{xx},$$

where $x \in \mathbb{R}$, $t > 0$, $\rho-$ density, $u-$ velocity, $\theta-$ temperature, $\bar{p} = \bar{p}(\rho, \theta)$ (equation of state) and $\lambda, \mu, \kappa > 0$, constants.

General structure

The (cNSF) system is a viscous system of conservation laws of the form

$$U_t + F(U)_x = \big(B(U)U_x\big)_x,$$

where

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho\big(e + \frac{1}{2}u^2\big) \end{pmatrix} \in \mathscr{U}, \quad (\text{“conserved” variables}),$$

$$F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + \bar{p} \\ \rho u(e + \frac{1}{2}u^2) + \bar{p}u \end{pmatrix}, \quad (\text{flux functions})$$

and appropriate viscosity tensor $B = B(U)$. This tensor has a kernel. It is positive semi-definite $B(U) \geq 0$.

Hyperbolicity of the inviscid system

Definition.

The "inviscid" system

$$U_t + F(U)_x = 0,$$

is hyperbolic if $A(U) := \partial_U F(U) \in \mathbb{R}^{n\times n}$ is diagonalizable over $\mathbb{R}$ (all eigenvalues are real and semi-simple) for all $U \in \mathscr{U} \subseteq \mathbb{R}^n$.

Motivation: It is a necessary condition for the inviscid system to be well-posed. Moreover, it guarantees traveling wave solutions of the form $U(x,t) = \varphi(x - st)$, $s \in \mathbb{R}$, yielding the eigenvalue problem

$$(A(\varphi(\cdot)) - sI)\varphi'(\cdot) = 0.$$

Genuine coupling (physical interpretation)

The viscosity tensor B strictly dissipates the underlying hyperbolic system provided that:

$$\textbf{No eigenvector of } A(U) \textbf{ lies in the kernel of } B(U), \quad \forall U \in \mathscr{U}\,.$$

This property is also known as **genuine coupling**.

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Heuristically: all "hyperbolic waves" are dissipated by the viscous terms.

Let $\bar{U} \in \mathscr{U}$ be fixed. Linearize the hyperbolic system around $\bar{U}$

$$U_t + A(\bar{U})U_x = 0.$$

Let $\lambda \in \sigma(A(\bar{U})) \subset \mathbb{R}$ with $A(\bar{U})V = \lambda V$, $V = V(\bar{U}) \in \mathbb{R}^n \Rightarrow$ planar wave of the form

$$\Phi(x,t) = \varphi(\theta(x,t))V, \quad \theta(x,t) = x - \lambda t, \quad \varphi \in C^2(\mathbb{R}),$$

Upon substitution into the linearized viscous system:

$$\varphi'(\theta)\big(-\lambda V + A(\bar{U})V\big) = \varphi''(\theta)B(\bar{U})V.$$

Friedrichs' symmetrizability

Definition

The system is symmetrizable in the sense of Friedrichs if there exists $A_0 = A_0(U)$, smooth, symmetric, positive definite, such that $A_0(U)A(U)$, and $A_0(U)B(U)$are symmetric for each $U \in \Omega$.

Note: It is well-known that

- Friedrichs symmetrizability $\Rightarrow$ hyperbolicity.
- If the system has conservative form (e.g. (cNSF)) then the symmetrizer is the Hessian of the entropy function, $A_0 = \partial_U^2 \eta$.

Linearization around a constant state

Generic viscous system in quasilinear form,

$$U_t + A(U)U_x = (B(U)U_x)_x, \tag{VS}$$

with $U \in \mathscr{U} \subseteq \mathbb{R}^n$.

If $\bar{U} + U$ is a solution to (VS), where $\bar{U} \in \mathscr{U}$ is a constant equilibrium state, then

$$U_t + \bar{A}U_x = \bar{B}U_{xx} + \widetilde{\mathscr{N}} = 0,$$

where $\bar{A} = A(\bar{U}), \bar{B} = B(\bar{U})$ and $\widetilde{\mathscr{N}}$ comprises nonlinear terms.

Assume the system is Friedrichs symmetrizable: there exists a constant, positive, symmetric matrix $A_0 > 0$ such that $A := A_0\bar{A}$, $B := A_0\bar{B}$ are symmetric. The linearized system is

$$A_0 U_t + AU_x = BU_{xx}.$$

Take the Laplace-Fourier transform to obtain the following spectral problem:

$$(\lambda A_0 + i\xi A + \xi^2 B)\hat{U} = 0.$$

Strict dissipativity and genuine coupling

Definition (strict dissipativity)

The linear system is said to be strictly dissipative if the solutions to the spectral problem satisfy

$$\operatorname{Re} \lambda(\xi) < 0,$$

for each $\xi \neq 0$.

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Definition (genuine coupling).

The system is genuinely coupled if no eigenvector of A is in $\ker B$.

Note: Genuine coupling is also known in the literature as the Kawashima-Shizuta condition or the K-condition.

Compensating matrix

Definition (compensating matrix)

K is a compensating matrix function of the linear system provided that:

(i) KA_0 is skew-symmetric.

(ii) $[KA]^s + B > 0$ is positive definite. Here $[M]^s = \frac{1}{2}(M + M^\top)$ denotes the symmetric part of M.

Kawashima-Shizuta theory (i): The equivalence theorem

S. Kawashima (Ph. D. thesis, Kyoto Univ., 1983) formulated the nonlinear decay structure of solutions in terms of the strict dissipativity of the linearized problem (see also **Kawashima and Shizuta, 1985, 1988**).

Kawashima-Shizuta theory (i): The equivalence theorem

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Equivalence theorem (Kawashima, Shizuta, 1985)
Under the hypotheses (symmetry, $A_0 > 0$, $B \geq 0$), the next statements are equivalent:

(a) The system is strictly dissipative.

(b) The system is genuinely coupled.

(c) The system admits a compensating matrix.

(d) There exists a constant $\delta > 0$ such that

$$\operatorname{Re} \lambda(\xi) \leq -\delta \frac{|\xi|^2}{1 + |\xi|^2}.$$

Kawashima-Shizuta theory (ii): Nonlinear results

- The dissipative structure of the system implies a nonlinear decay structure for solutions in a neighborhood of constant states:
  - perturbations of constant equilibrium states (**Kasashima, Shizuta, 1988**).
  - small-amplitude shock profiles (**Humpherys, Zumbrun, 2002**).
  - solutions relaxing to Maxwellian states (**Yong, 2009**).
- The decay estimates can be performed at the lowest level and depend upon the properties of the compensating matrix K.
- The (small) nonlinear terms can be handled via local existence theorems and absorbed into the linear energy estimates.

Humpherys' extension and the equivalence theorem

Extension to higher order systems

Humpherys (2005) extended the notion of strict dissipativity to higher order systems of the form

$$U_t = \mathscr{L} U := -\sum_{k=0}^{m} D_k \partial_x^k U,$$

with $U \in \mathbb{R}^n$, $m \in \mathbb{Z}$, $m \geq 1$, $D_k \in \mathbb{R}^{n \times n}$ constant matrices.

Fourier decomposition

Taking Fourier-Laplace transform yields

$$(\lambda + i\xi A(\xi) + \xi^2 B(\xi))\hat{U} = 0,$$

where Humpherys distinguishes between odd and even symbols:

$$A(\xi) := \sum_{k \text{ odd}} (i\xi)^{k-1} D_k, \quad B(\xi) := \sum_{k \text{ even}} (-1)^{k/2} \xi^{k-2} D_k.$$

Strict dissipativity and genuine coupling

Definition.

- The operator $\mathscr{L}$ is called strictly dissipative if for each $\xi \neq 0$ there holds $\operatorname{Re} \lambda(\xi) < 0$.
- $\mathscr{L}$ is called genuinely coupled if no eigenvector of $A(\xi)$ is in $\ker B(\xi)$ for all fixed $\xi \neq 0$.

Compensatng matrix

Definition (compensating matrix).

Let A_0, A, B be smooth, real matrix-valued functions of $\xi \in \mathbb{R}$. Assume that A_0, A, B are symmetric, with $A_0 > 0$, $B \geq 0$. A real matrix valued function $K = K(\xi)$, $K \in C^\infty(\mathbb{R};\mathbb{R}^{n\times n})$ is said to be a compensating function for the triplet (A_0, A, B) provided that:

(i) $K(\xi)A_0(\xi)$ is skew-symmetric for all $\xi \in \mathbb{R}$.

(ii) $[K(\xi)A(\xi)]^s + B(\xi) \geq \gamma I > 0$ for all $\xi \in \mathbb{R}$, $\xi \neq 0$, $\gamma > 0$.

Symbol symmetrizability

Humpherys extends the notion of Friedrichs'symmetrizability to that of symbol symmetrizability.

Definition.

The operator $\mathscr{L}$ is called symbol symmetrizable if there exists a symmetric, smooth, positive definite, real matrix-valued function $A_0(\xi) > 0$ such that both $A_0(\xi)A(\xi)$ and $A_0(\xi)B(\xi)$ are symmetric and $A_0(\xi)B(\xi) \geq 0$ (positive semi-definite).

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Observation: Not all systems which are symbol symmetrizable are Friedrichs' symmetrizable. Example: Korteweg system.

Equivalence theorem

Theorem (Humpherys, 2005)

Assuming $A_0(\xi)$ is a symbol symmetrizer for $\mathscr{L}$. Then the following statements are equivalent:

(a) $\mathscr{L}$ is strictly dissipative.

(b) $\mathscr{L}$ is genuinely coupled.

(c) There exists a compensating symbol $K(\xi)$ for the triplet $(A_0(\xi), A_0(\xi)A(\xi), A_0(\xi)B(\xi))$.

(d) There exists a constant $\delta > 0$ such that

$$\operatorname{Re}\lambda(\xi) \leq -\delta \frac{|\xi|^2}{1+|\xi|^2}.$$

Comments on Humpherys' extension

- Humpherys provides an explicit formula for the compensating matrix symbol $K(\xi)$.
- It is the Drazin's inverse of the commutator operator: $\text{ad}_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, $\text{ad}_A(B) = [A, B] := AB - BA$.
  - $A \in M_n(\mathbb{C})$ semi-simple, with distinct eigenvalues $\{\lambda_j\}_{j=1}^r$, $1 \leq r \leq n$, multiplicities $m_j \in \mathbb{N}$, $\sum_{j=1}^r m_j = n$. Eigenprojections $P_j = \frac{1}{2\pi i} \int_{\Gamma_j} (z - A)^{-1} \, dz$.
  - Define $K(B) := \sum_{i \neq j} \frac{P_i B P_j}{\lambda_i - \lambda_j}$ for each $B \in M_n$.
  - Then $\text{ad}_A(K(B)) = B$ and $K = B(\xi))$ is the compensating matrix symbol.
- However, there are degrees of freedom to choose the compensating matrix symbol.
- The pointwise estimate in (d) can be improved by providing an expression of K by inspection.

Example: compressible fluids of Korteweg type

Example: compressible fluids of Korteweg type

One-dimensional system for isothermal compressible fluids of Korteweg type in Lagrangian coordinates:

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + p(v)_x = \Big(\frac{\mu(v)}{v} u_x\Big)_x - \Big(\kappa(v) v_{xx} + \tfrac{1}{2}\kappa'(v) v_x^2\Big)_x, \end{aligned} \tag{iK}$$

where $x \in \mathbb{R}$, $t > 0$, $u-$ velocity, $v-$ specific volume. μ, κ viscosity and capillarity coefficients. $p = p(v)$, pressure. Here $p'(v) < 0$ for all $v \in \mathscr{D}_0 = \{C_0^{-1} < v < C_0\}$. It is assumed that p, κ, μ smooth and uniformly positive.

Nonlinear system for perturbations of equilibrium states

Consider a constant equilibrium state $(\bar{u}, \bar{v}) \in \mathbb{R} \times \mathscr{D}_0$. If $u + \bar{u}$, $v + \bar{v}$ are solutions (with now u, v perturbations), we have:

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(\bar{v} + v)_x &= \Big(\frac{\mu(\bar{v} + v)}{\bar{v} + v} u_x \Big)_x - \Big(\kappa(\bar{v} + v) v_{xx} + \tfrac{1}{2} \kappa'(\bar{v} + v) v_x^2 \Big)_x \end{aligned} \qquad \text{(pK)}$$

Local existence (i)

The local existence of solutions to system (iK) is well understood (see **Hatori and Li (1994)**, **Danchin and Dejardins (2001)**, **Chen et al. (2015)**.

For $M \geq m > 0$, $T > 0$, any $s \geq 3$, denote the function space

$$
\begin{aligned}
X_s((0,T);m,M) := \Big\{(v,u) :\ & v - \bar{v} \in C((0,T);H^{s+1}(\mathbb{R})) \cap C^1((0,T);H^{s-1}(\mathbb{R})), \\
& u - \bar{u} \in C((0,T);H^{s}(\mathbb{R})) \cap C^1((0,T);H^{s-2}(\mathbb{R})), \\
& (v_x, u_x) \in L^2((0,T);H^{s+1}(\mathbb{R}) \times H^{s}(\mathbb{R})), \\
& \text{and } m \leq v(x,t) \leq M \text{ a.e. in } (x,t) \in \mathbb{R} \times (0,T)\Big\}.
\end{aligned}
$$

Local existence (ii)

For $U := (v+\bar{v}, u+\bar{u})^\top \in X_s((0,T);m,M)$ and any $0 \le t_1 \le t_2 \le T$ we define

$$|||U|||_{s,[t_1,t_2]}^2 := \sup_{t_1 \le t \le t_2} \left(\|v(t)\|_{s+1}^2 + \|u(t)\|_s^2 \right) + \int_{t_1}^{t_2} \left(\|v_x(t)\|_{s+1}^2 + \|u_x(t)\|_s^2 \right) dt,$$

$$|||U|||_{s,T} := |||U|||_{s,[0,T]}.$$

Local existence (iii)

Theorem. (Hattori, Li, 1994)

Under the assumptions, suppose $(v_0, u_0) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 3$. Then we can find $a_0 > 0$ constant such that if

$$\left(\|v_0\|_{s+1}^2 + \|u_0\|_s^2\right)^{1/2} \leq a_0,$$

then there holds $0 < m_0 \leq \bar{v} + v_0(x) \leq M_0$ a.e. in $x \in \mathbb{R}$ for some $M_0 \geq m_0 > 0$, and there exists $T_1 = T_1(a_0)$ such that a unique smooth local solution $U = (v + \bar{v}, u + \bar{u})^\top \in X_s\left((0, T_1); \frac{1}{2}m_0, 2M_0\right)$ exists for the Cauchy problem (pK) with initial condition $U(0) = (v_0 + \bar{v}, u_0 + \bar{u})^\top$. Moreover,

$$|\!|\!| U |\!|\!|_{s, T_1} \leq C_0 \left(\|v_0\|_{s+1}^2 + \|u_0\|_s^2\right)^{1/2},$$

for some constant $C_0 > 1$ depending only on a_0.

Strict dissipativity of the Korteweg system (i)

Linearize system (pK) around $(\bar{u}, \bar{v})$, $\bar{v} > 0$, to obtain:

$$\begin{aligned} v_t - u_x &= 0, \\ u_t - \bar{q} v_x &= \frac{\bar{\mu}}{\bar{v}} u_{xx} - \bar{\kappa} v_{xxx}, \end{aligned}$$

where $\bar{q} := -p'(\bar{v}) > 0$, $\bar{\mu} := \mu(\bar{v}) > 0$, $\bar{\kappa} := \kappa(\bar{v}) > 0$. Following Humpherys, write the linear system as

$$U_t = \mathscr{L} U = -\sum_{k=0}^{3} D_k \partial_x^k U,$$

where

$$U = \begin{pmatrix} v \\ u \end{pmatrix}, D_1 = -\begin{pmatrix} 0 & 1 \\ \bar{q} & 0 \end{pmatrix}, D_2 = -\begin{pmatrix} 0 & 0 \\ 0 & \bar{\mu}/\bar{v} \end{pmatrix}, D_3 = \begin{pmatrix} 0 & 0 \\ \bar{\kappa} & 0 \end{pmatrix}, D_0 \equiv 0.$$

Strict dissipativity of the Korteweg system (ii)

The corresponding symbols are

$$\widetilde{A}(\xi) = D_1 - \xi^2 D_3 = \begin{pmatrix} 0 & -1 \\ -(\bar{q} + \xi^2 \bar{\kappa}) & 0 \end{pmatrix},$$

$$\widetilde{B}(\xi) = -\xi^2 D_2 = \xi^2 \begin{pmatrix} 0 & 0 \\ 0 & \bar{\mu}/\bar{v} \end{pmatrix}.$$

Taking Fourier transform we obtain the linear system

$$\partial_t \hat{U} + i\xi \widetilde{A}(\xi)\hat{U} + \widetilde{B}(\xi)\hat{U} = 0.$$

Symbol smmetrizability

Lemma.

The Korteweg operator $\mathscr{L}$ is symbol symmetrizable. The symmetrizer can be chosen as

$$A_0(\xi) := \begin{pmatrix} \beta(\xi) & 0 \\ 0 & 1 \end{pmatrix}.$$

with $\beta(\xi) := \bar{q} + \xi^2 \bar{\kappa} > 0$. Here A_0 is uniformly bounded, symmetric, positive-definite.

Proof. Follows by direct computations.

□

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Proof. Follows by direct computations.

$\square$

Note: The Korteweg system is **not** symmetrizable in the sense of Friedrichs.

Symmetric system

Symmetric linear system:

$$A_0 \hat{U}_t + i\xi A_0 \widetilde{A} \hat{U} + \xi^2 A_0 \widetilde{B} \hat{U} = 0.$$

with

$$A_0(\xi)\widetilde{A}(\xi) = -\beta(\xi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$A_0(\xi)\widetilde{B}(\xi) = \xi^2 \begin{pmatrix} 0 & 0 \\ 0 & \bar{\mu}/\bar{v} \end{pmatrix} =: \xi^2 \bar{B} \geq 0.$$

Genuine coupling (i)

Lemma.

The linearized Korteweg operator satisfies the genuine coupling condition. Moreover, the matrix symbol $A_0(\xi)\widetilde{A}(\xi)$ is of constant multiplicity in ξ.

Proof. Follows by direct computation.

□

Genuine coupling (ii)

Change of variables:

$$\hat{V} := A_0(\xi)^{1/2}\hat{U},$$

so that the linear equation is now

$$\hat{V}_t + i\xi A(\xi)\hat{V} + B(\xi)\hat{V} = 0, \qquad \text{(L)}$$

with

$$A(\xi) := A_0(\xi)^{1/2}A(\xi)A_0(\xi)^{-1/2} = \begin{pmatrix} 0 & -\beta(\xi)^{1/2} \\ -\beta(\xi)^{1/2} & 0 \end{pmatrix},$$

$$B(\xi) := A_0(\xi)^{1/2}B(\xi)A_0(\xi)^{-1/2} = \xi^2 \begin{pmatrix} 0 & 0 \\ 0 & \bar{\mu}/\bar{v} \end{pmatrix} = \xi^2 \bar{B}.$$

Compensating matrix

Lemma.

Let us define

$$K(\xi) := \frac{\bar{\mu}}{4\beta(\xi)^{1/2}\bar{v}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then:

- $K \in C^\infty(\mathbb{R})$
- $K = K(\xi)$ is a compensating matrix function for the triplet (I, A, B).
- $|K|, |\xi K|, |K\hat{B}^{1/2}|$ are uniformly bounded in $\xi \in \mathbb{R}$.

Proof. By inspection.

□

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Proof. By inspection.

□

By the equivalence theorem this guarantees genuine coupling.

Basic energy estimate

Lemma (basic energy estimate).

The solutions $\hat{V} = \hat{V}(\xi, t)$ to system (L) satisfy the estimate

$$|\hat{V}(\xi, t)| \leq C \exp(-k\xi^2 t)|\hat{V}(\xi, 0)|,$$

for all $\xi \in \mathbb{R}$, $t \geq 0$ and some uniform constants $C, k > 0$.

Proof of Lemma

Proof. Take the real part of the $\mathbb{C}^2$-inner product of $\hat{V}$ with equation (L).

$$\tfrac{1}{2}\partial_t|\hat{V}|^2 + \xi^2\langle \hat{V}, \bar{B}\hat{V}\rangle = 0, \qquad (*)$$

in view that $B(\xi) = \xi^2\bar{B}$.

Multiply (L) by $-i\xi K$ and take the inner product with $\hat{V}$:

$$-\langle \hat{V}, i\xi K\hat{V}_t\rangle + \xi^2\langle \hat{V}, KA\hat{V}\rangle - \langle \hat{V}, i\xi^3 K\bar{B}\hat{V}\rangle = 0.$$

Since K is skew-symmetric we have

$$\mathrm{Re}\,\langle \hat{V}, i\xi K\hat{V}_t\rangle = \tfrac{1}{2}\xi\partial_t\langle \hat{V}, iK\hat{V}\rangle.$$

Taking the real part of the previous equation yields

$$-\frac{1}{2}\xi\partial_t\langle\hat{V}, iK\hat{V}\rangle + \xi^2\langle\hat{V}, [KA]^s\hat{V}\rangle = \text{Re}\left(i\xi^3\langle\hat{V}, K\bar{B}\hat{V}\rangle\right).$$

Use $\bar{B} \geq 0$, $[KA]^s = KA$ symmetric and $\xi K(\xi)$ is uniformly bounded in ξ. We arrive at

$$-\frac{1}{2}\xi\partial_t\langle\hat{V}, iK\hat{V}\rangle + \xi^2\langle\hat{V}, KA\hat{V}\rangle \leq \varepsilon\xi^2|\hat{V}|^2 + C_\varepsilon\xi^2\langle\hat{V}, \bar{B}\hat{V}\rangle \qquad (**)$$

for any $\varepsilon > 0$, where $C_\varepsilon > 0$ depends only on $\varepsilon > 0$, $|\bar{B}^{1/2}|$.

Now multiply equation (**) by $\delta > 0$ and add it to (*) to obtain,

$$\frac{1}{2}\partial_t[|\hat{V}|^2 - \delta\xi\langle\hat{V}, iK\hat{V}\rangle] + \xi^2[\delta\langle\hat{V}, KA\hat{V}\rangle + (1-\delta C_\varepsilon)\langle\hat{V}, \bar{B}\hat{V}\rangle] \leq \varepsilon\delta\xi^2|\hat{V}|^2.$$

Let us define the energy,

$$\mathscr{E} := |\hat{V}|^2 - \delta\xi\langle \hat{V}, iK\hat{V}\rangle.$$

$\mathscr{E}$ is real because K is skew-symmetric. Moreover, in view that for $|\xi A| \leq C$ uniformly for all ξ, there exists $\delta_0 > 0$ sufficiently small such that

$$C_1^{-1}|\hat{V}|^2 \leq \mathscr{E} \leq C_1|\hat{V}|^2,$$

for some $C_1 > 0$, provided that $0 < \delta < \delta_0$. Hence, $\mathscr{E}$ is indeed an energy, equivalent to $|\hat{V}|^2$, for $\delta > 0$ sufficiently small.

Now from property of the compensating function K there exists $\theta > 0$ such that $\langle \hat{V}, ([KA]^s + B)\hat{V} \rangle \geq \theta |\hat{V}|^2$. Therefore, by taking $\varepsilon = \theta/2$, $0 < \delta < \delta_0$ small enough such that $(1 + C_\varepsilon)\delta < 1$, we have

$$\delta \langle \hat{V}, KA\hat{V} \rangle + (1 - \delta C_\varepsilon) \langle \hat{V}, \bar{B}\hat{V} \rangle \geq \delta \overline{\theta} |\hat{V}|^2.$$

Then we arrive at

$$\tfrac{1}{2}\partial_t \mathscr{E} + \tfrac{1}{2}\xi^2 \delta\theta |\hat{V}|^2 \leq 0,$$

and obtain

$$\partial_t \mathscr{E} + k\xi^2 \mathscr{E} \leq 0,$$

where $k = \delta\theta / C_1 > 0$. This yields the result.

$\square$

Corollary.

The solutions $\hat{U}(\xi,t) = \big(\hat{U}_1(\xi,t), \hat{U}_2(\xi,t)\big)^\top$ to the linear system satisfy the estimate

$$(1+\xi^2)|\hat{U}_1(\xi,t)|^2 + |\hat{U}_2(\xi,t)|^2 \leq C\exp(-2k\xi^2 t)\big[(1+\xi^2)|\hat{U}_1(\xi,0)|^2 + |\hat{U}_2(\xi,0)|^2\big],$$

for all $\xi \in \mathbb{R}$, $t \geq 0$ and some uniform constants $C > 0$ and $k > 0$.

Lemma.

Suppose that $U = (U_1, U_2)^\top$ is a solution to the linear evolution system $U_t = \mathscr{L} U$ with initial data

$$U(x,0) \in \big(H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\big) \cap \big(L^1(\mathbb{R}) \times L^1(\mathbb{R})\big), \quad s \geq 2.$$

Then for each fixed $0 \leq \ell \leq s$, $\ell \in \mathbb{Z}$, there holds the estimate,

$$\Big(\|\partial_x^\ell U_1(t)\|_1^2 + \|\partial_x^\ell U_2(t)\|_0^2\Big)^{1/2} \leq C e^{-c_1 t}\Big(\|\partial_x^\ell U_1(0)\|_1^2 + \|\partial_x^\ell U_2(0)\|_0^2\Big)^{1/2} + \\ + C(1+t)^{-(\ell/2+1/4)} \|U(0)\|_{L^1},$$

for all $t \geq 0$ and some uniform constants $C, c_1 > 0$.

Proof

Proof. Fix $\ell \in [0, s]$, multiply by $\xi^{2\ell}$ and integrate in $\xi \in \mathbb{R}$:

$$\int_{\mathbb{R}} \left[\xi^{2\ell}(1+\xi^2)|\hat{U}_1(\xi,t)|^2 + \xi^{2\ell}|\hat{U}_2(\xi,t)|^2\right] d\xi \le CJ_1(t) + CJ_2(t),$$

where,

$$J_1(t) := \int_{-1}^{1} \left[\xi^{2\ell}(1+\xi^2)|\hat{U}_1(\xi,0)|^2 + \xi^{2\ell}|\hat{U}_2(\xi,0)|^2\right] \exp(-2k\xi^2 t)\, d\xi,$$

$$J_2(t) := \int_{|\xi|\ge 1} \left[\xi^{2\ell}(1+\xi^2)|\hat{U}_1(\xi,0)|^2 + \xi^{2\ell}|\hat{U}_2(\xi,0)|^2\right] \exp(-2k\xi^2 t)\, d\xi.$$

Clearly,

$$J_1(t) \le 2 \sup_{\xi \in \mathbb{R}} |\hat{U}(\xi,0)|^2 \int_{-1}^{1} \xi^{2\ell} e^{-kt\xi^2}\, d\xi.$$

For any fixed $\ell \in [0, s]$, $k > 0$, the integral

$$I_0(t) := (1+t)^{\ell+1/2} \int_{-1}^{1} \xi^{2\ell} e^{-kt\xi^2} \, d\xi \leq C,$$

is uniformly bounded for all $t > 0$: $I_0(t) \leq C$, with some constant $C > 0$ (see auxiliary lemma). Hence,

$$J_1(t) \leq C(1+t)^{-(\ell+1/2)} \|U(x,0)\|_{L^1}^2.$$

On the other hand if $|\xi| \geq 1$ then $\exp(-2kt\xi^2) \leq \exp(-kt)$. Therefore, from Plancherel's theorem,

$$\begin{aligned}
J_2(t) &\leq e^{-kt} \int_{|\xi|\geq 1} \xi^{2\ell}(1+\xi^2)|\hat{U}_1(\xi,0)|^2 + \xi^{2\ell}|\hat{U}_2(\xi,0)|^2 \, d\xi \\
&= e^{-kt} \int_{|\xi|\geq 1} (\xi^{2\ell} + \xi^{2(\ell+1)})|\hat{U}_1(\xi,0)|^2 + \xi^{2\ell}|\hat{U}_2(\xi,0)|^2 \, d\xi \\
&\leq e^{-kt} \int_{\mathbb{R}} (\xi^{2\ell} + \xi^{2(\ell+1)})|\hat{U}_1(\xi,0)|^2 + \xi^{2\ell}|\hat{U}_2(\xi,0)|^2 \, d\xi \\
&= e^{-kt} \big(\|\partial_x^\ell U_1(0)\|_1^2 + \|\partial_x^\ell U_2(0)\|_0^2 \big),
\end{aligned}$$

for all $t > 0$ and if $\ell \in \mathbb{Z}$. Combining both estimates we obtain the result with $c_1 = k/2 > 0$ and the lemma is proved.

□

Semigroup decay

The semigroup associated to $U_t = \mathscr{L} U$ is

$$(e^{t\mathscr{L}} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} e^{tM(i\xi)} \hat{f}(\xi)\, d\xi,$$

$$M(z) := \begin{pmatrix} 0 & z \\ z(\bar{q} - z^2 \bar{\kappa}) & z^2 \bar{\mu}/\bar{v} \end{pmatrix}, \quad z \in \mathbb{C},$$

$$M(i\xi) = -(i\xi A(\xi) + \xi^2 \bar{B}), \qquad \xi \in \mathbb{R},$$

and $U(x,t) = (e^{t\mathscr{L}} f)(x)$ is the solution with initial condition $f = (f_1, f_2)^\top$.

Corollary (semigroup decay).

For any $f \in \big(H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\big) \cap \big(L^1(\mathbb{R}) \times L^1(\mathbb{R})\big)$, $s \geq 2$, and all $0 \leq \ell \leq s$, $t > 0$, there holds

$$\Big(\|\partial_x^\ell (e^{t\mathscr{L}} f)_1(t)\|_1^2 + \|\partial_x^\ell (e^{t\mathscr{L}} f)_2(t)\|_0^2 \Big)^{1/2} \leq C e^{-c_1 t} \Big(\|\partial_x^\ell f_1\|_1^2 + \|\partial_x^\ell f_2\|_0^2 \Big)^{1/2} + \\ + C(1+t)^{-(\ell/2+1/4)} \|f\|_{L^1},$$

for some uniform $C, c_1 > 0$.

Nonlinear system

The nonlinear system for the perturbations $U = (u, v)$,

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(\bar{v}+v)_x &= \Big(\frac{\mu(\bar{v}+v)}{\bar{v}+v} u_x\Big)_x - \Big(\kappa(\bar{v}+v) v_{xx} + \tfrac{1}{2}\kappa'(\bar{v}+v) v_x^2\Big)_x \end{aligned} \tag{pK}$$

can be recast as

$$U_t = \mathscr{L} U + \partial_x H,$$

where the (conservative) nonlinear terms are

$$H(U, U_x, U_{xx}) = \begin{pmatrix} 0 \\ H_2(U, U_x, U_{xx}) \end{pmatrix},$$

$$\begin{aligned} H_2(U, U_x, U_{xx}) := -(p(\bar{v}+v) - p(\bar{v}) - p'(\bar{v})v) + \Big(\frac{\mu(\bar{v}+v)}{\bar{v}+v} u_x - \frac{\mu(\bar{v})}{\bar{v}} u_x\Big) + \\ + (\kappa(\bar{v}+v) v_{xx} - \kappa(\bar{v}) v_{xx}) + \tfrac{1}{2}\kappa'(\bar{v}+v) v_x^2 \\ = O(v^2 + |v||u_x| + |v||v_{xx}| + v_x^2). \end{aligned}$$

Duhamel's principle

Initial condition

$$U(x,0) = U_0(x) := (v_0(x) + \bar{v}, u_0(x) + \bar{u})^\top .$$

Under the assumptions of the local existence theorem, suppose

$$(v_0, u_0)^\top \in \left(H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\right) \cap \left(L^1(\mathbb{R}) \times L^1(\mathbb{R})\right),$$

for some $s \geq 3$. Using the semigroup, the local solution, $U = (U_1, U_2)^\top = (v + \bar{v}, u + \bar{u})^\top$ to (pK) is given by Duhamel's principle,

$$U(x,t) = e^{t\mathscr{L}} U_0(x) + \int_0^t e^{(t-z)\mathscr{L}} (H(x,z)_x)\, dz,$$

where $H(x,z) = (0, H_2(x,z))^\top$.

Use the semigroup estimates to obtain

$$\left(\|\partial_x^\ell v(t)\|_1^2 + \|\partial_x^\ell u(t)\|_0^2\right)^{1/2} \leq$$

$$\leq C\left(e^{-c_1 t}\left(\|\partial_x^\ell v_0\|_1^2 + \|\partial_x^\ell u_0\|^2\right)^{1/2} + (1+t)^{-(1/4+\ell/2)}\|(v_0,u_0)\|_{L^1}\right) +$$

$$+ C\int_0^t \left(e^{-c_1(t-z)}\|\partial_x^{\ell+1} H_2(\cdot,z)\|_0 + (1+t-z)^{-(3/4+\ell/2)}\|H_2(\cdot,z)\|_{L^1}\right) dz.$$

Notation. Define,

$$\|U(t)\|_k := \left(\|v(t)\|_{k+1}^2 + \|u(t)\|_k^2\right)^{1/2},$$

for all $0 \leq k \leq s$, and $\|U_0\|_{L^1} = \|(v_0,u_0)\|_{L^1}$.

Suming up for $\ell = 0, 1, \ldots, s-1$:

$$\|U(t)\|_{s-1} \leq C(1+t)^{-1/4}(\|U_0\|_{s-1} + \|U_0\|_{L^1}) + \\ + \int_0^t \left(e^{-c_1(t-z)}\|H_2(\cdot,z)\|_s + (1+t-z)^{-3/4}\|H_2(\cdot,z)\|_{L^1}\right) dz. \qquad (\otimes)$$

Need to estimate the nonlinear terms $\|H_2(\cdot,z)\|_s$ and $\|H_2(\cdot,z)\|_{L^1}$.

Estimates on the nonlinear terms

Lemma.

$$\begin{aligned}\|H_2(\cdot,z)\|_s &\leq C\big(\|v(z)\|_s^2 + \|v(z)\|_s\|v_{xx}(z)\|_s + \|v(z)\|_s\|u_x(z)\|_s + \|v(z)\|_2\|v_x(z)\|_{s+1}\big)\\ &\leq C\left(\|v(z)\|_s^2 + \|v(z)\|_s\|u_x(z)\|_s + \|v(z)\|_s\|v_x(z)\|_{s+1}\right),\end{aligned}$$

$$\|H_2(\cdot,z)\|_{L^1} \leq C\|(v,u)(z)\|_2^2 \leq C\|U(z)\|_2^2 \leq C\|U(z)\|_{s-1}^2,$$

for all $z \in [0,t]$, $s \geq 3$, $C > 0$ uniform constant.

Proof. By Sobolev embedding and Banach algebra inequalities (Sobolev calculus).

□

Upon substitution into $(\otimes)$:

$$
\begin{aligned}
\|U(t)\|_{s-1} \le C\,(1+t)^{-1/4}\,(\|U_0\|_{s-1} + \|U_0\|_{L^1}) +& \\
&+ C \sup_{0\le z\le t} \|v(z)\|_s \int_0^t e^{-c_1(t-z)} \|v(z)\|_s\, dz + \\
&+ C \left(\int_0^t \|u_x(z)\|_s^2\, dz \right)^{1/2} \left(\int_0^t e^{-2c_1(t-z)} \|v(z)\|_s^2\, dz \right)^{1/2} + \\
&+ C \left(\int_0^t \|v_x(z)\|_{s+1}^2\, dz \right)^{1/2} \left(\int_0^t e^{-2c_1(t-z)} \|v(z)\|_s^2\, dz \right)^{1/2} + \\
&+ C \int_0^t (1+t-z)^{-3/4}\, \|U(z)\|_{s-1}^2\, dz.
\end{aligned}
$$

Define,

$$
E_s(t) := \sup_{0\le z\le t} (1+z)^{1/4}\, \|U(z)\|_{s-1}.
$$

Hence,

$$E_s(t) := \sup_{0\le z\le t} (1+z)^{1/4} \|U(z)\|_{s-1}.$$

Hence, we obtain

$$E_s(t) \le C\big(\|U_0\|_{s-1} + \|U_0\|_{L^1}\big) + CI_1(t)|\!|\!|U|\!|\!|_{s,t} E_s(t) + CI_2(t)E_s(t)^2, \quad \text{(E)}$$

where

$$I_1(t) := \sup_{0\le z\le t} (1+z)^{1/4} \int_0^z e^{-c_1(z-z_1)}(1+z_1)^{-1/4}\, dz_1 + \\ + \sup_{0\le z\le t} (1+z)^{1/4} \left[\int_0^z e^{-2c_1(z-z_1)}(1+z_1)^{-1/2}\, dz_1\right]^{1/2},$$

$$I_2(t) := \sup_{0\le z\le t} (1+z)^{1/4} \int_0^z (1+z-z_1)^{-3/4} (1+z_1)^{-1/2}\, dz_1.$$

Auxiliary lemma.

There exists a uniform constant $C>0$ such that

$$I_0(t)=(1+t)^{\ell+1/2}\int_{-1}^{1}\xi^{2\ell}e^{-kt\xi^2}\,d\xi\le C,$$

$$I_1(t)=\sup_{0\le z\le t}(1+z)^{1/4}\int_0^z e^{-c_1(z-\tau)}(1+\tau)^{-1/4}\,d\tau+$$

$$+\sup_{0\le z\le t}(1+z)^{1/4}\left[\int_0^z e^{-2c_1(z-\tau)}(1+\tau)^{-1/2}\,d\tau\right]^{1/2}\le C,$$

$$I_2(t)=\sup_{0\le z\le t}(1+z)^{1/4}\int_0^z(1+z-\tau)^{-3/4}(1+\tau)^{-1/2}\,d\tau\le C$$

for all $t\ge 0$ and fixed $0\le\ell\le s-1$, $s\ge 2$.

Proof. I_1 and I_0 by standard calculus tools.

I_2 is an elliptic integral:

$$A_2(z) = (1+z)^{1/4} \int_0^z (1+z-\tau)^{-3/4} (1+\tau)^{-1/2}\, d\tau \to 4 \int_0^1 \frac{dy}{\sqrt{1-y^2}\sqrt{1+y^2}} = 4F(\tfrac{\pi}{2}\,|\,-1)$$

as $z \to \infty$, where F is the incomplete elliptic integral of the first kind,

$$F(\varphi\,|\,k) = \int_0^{\sin\varphi} \frac{dy}{\sqrt{1-y^2}\sqrt{1-ky^2}}.$$

$A_2(z)$ has a finite limit when $z \to \infty$ ($A_2 \to 4F(\frac{\pi}{2}\,|\,-1) \approx 5{,}2441$), $A_2(z)$ is continuous in any compact interval $z \in [0, R]$.

□

Consequence: since both $I_1(t)$ and $I_2(t)$ are uniformly bounded, we readily obtain the energy estimate

$$E_s(t) \leq C\big(\|U_0\|_{s-1} + \|U_0\|_{L^1}\big) + C|||U|||_{s,t} E_s(t) + CE_s(t)^2. \qquad \text{(EE)}$$

Global existence and asymptotic decay

Theorem (P., Valdovinos, 2022)

Under the hypotheses, assume

$$(v_0, u_0)^\top \in \left(H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\right) \cap \left(L^1(\mathbb{R}) \times L^1(\mathbb{R})\right), \quad s \geq 3.$$

There exists $\delta_2 > 0$ such that if $\|U_0\|_s + \|U_0\|_{L^1} \leq \delta_2$, then the Cauchy problem for the nonlinear system (pK) has a unique global solution $(v+\bar{v}, u+\bar{u})(x,t)$ satisfying

$$\begin{aligned}
&v \in C\left((0,\infty); H^{s+1}(\mathbb{R})\right) \cap C^1((0,\infty); H^{s-1}(\mathbb{R})),\\
&u \in C\left((0,\infty); H^{s}(\mathbb{R})\right) \cap C^1((0,\infty); H^{s-2}(\mathbb{R}))\\
&(v_x, u_x) \in L^2\left((0,\infty); H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\right).
\end{aligned}$$

Furthermore, the solution satisfies

$$|||U|||_{s,t} \leq C_2 \|U_0\|_s,$$

$$\|U(t)\|_{s-1} \leq C_1 (1+t)^{-1/4} (\|U_0\|_{s-1} + \|U_0\|_{L^1}),$$

for every $t \in [0,\infty)$.

Proof sketch. Follows by standard nonlinear iteration and application of the energy estimate (EE).

□

Details in

- **P, Valdovinos**, J. Math. Anal. Appl. 514 (2022), no. 2, 126336.

Proof sketch. Follows by standard nonlinear iteration and application of the energy estimate (EE).

□

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Note: **Kawashima et al.**, Comm. PDE 47 (2022), no. 2, 378–400 studied the linear decay rate for the Korteweg system with energy exchange, partial Friedrichs' symmetrizability, estimates at the linear level.

Further developments

Korteweg model with energy exchange

One dimensional Navier-Stokes-Fourier-Korteweg system,

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x\big(\rho u^2 + p\big) &= \partial_x\big(\mu u_x + K\big), \\ \partial_t\big(\rho\varepsilon + \tfrac{1}{2}\rho u^2\big) + \partial_x\big(\rho u\big(\varepsilon + \tfrac{1}{2}u^2\big) + pu\big) &= \partial_x\big(\alpha\theta_x + \mu u u_x + uK + w\big), \end{aligned} \tag{θK}$$

where the Korteweg stress tensor, K, and the interstitial work, w, are

$$\begin{aligned} K &= \kappa\rho\rho_{xx} + \rho\kappa_x\rho_x - \tfrac{1}{2}\kappa_\rho\rho\rho_x^2 - \tfrac{1}{2}\kappa\rho_x^2, \\ w &= -\kappa\rho\rho_x u_x. \end{aligned}$$

Note: Here we use Eulerian coordinates, $v = 1/\rho$.

The linear (Fourier) system has the form

$$\hat{U}_t + i\xi \widetilde{A}(\xi)\hat{U} + \xi^2 \widetilde{B}\hat{U} = 0,$$

with symbol symmetrizer,

$$S_0(\xi) = \begin{pmatrix} \beta(\xi)/\bar{p}_\rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in C^\infty\big(\mathbb{R};\mathbb{R}^{3\times 3}\big).$$

where $\beta(\xi) := \bar{p}_\rho + \xi^2 \bar{\kappa}\bar{\rho} > 0$, $\hat{U} := S_0(\xi)^{1/2}\big(A^0\big)^{1/2}(\hat{\rho}, \hat{u}, \hat{\theta})^\top$, $A(\xi) = A^1 + \xi^2 C$,

$$\widetilde{A}(\xi) := S_0(\xi)^{1/2}\big(A^0\big)^{-1/2} A(\xi)\big(S_0(\xi)^{1/2}\big(A^0\big)^{1/2}\big)^{-1},$$
$$\widetilde{B}(\xi) := S_0(\xi)^{1/2}\big(A^0\big)^{-1/2} B\big(S_0(\xi)^{1/2}\big(A^0\big)^{1/2}\big)^{-1}.$$

$$A^0 = \begin{pmatrix} \bar{p}_\rho / \bar{\rho}\bar{\theta} & 0 & 0 \\ 0 & \bar{\rho}/\bar{\theta} & 0 \\ 0 & 0 & \bar{\varepsilon}_\theta \bar{\rho}/\bar{\theta}^2 \end{pmatrix}, \quad A^1 = \begin{pmatrix} \bar{p}_\rho \bar{u}/\bar{\theta}\bar{\rho} & \bar{p}_\rho/\bar{\theta} & 0 \\ \bar{p}_\rho/\bar{\theta} & \bar{u}\,\bar{\rho}/\bar{\theta} & \bar{p}_\theta/\bar{\theta} \\ 0 & \bar{p}_\theta/\bar{\theta} & \bar{\rho}\,\bar{\varepsilon}_\theta \bar{u}/\bar{\theta}^2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{\mu}/\bar{\theta} & 0 \\ 0 & 0 & \bar{\alpha}/\bar{\theta}^2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ \bar{\kappa}\bar{\rho}/\bar{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$A^0 = \begin{pmatrix} \bar{p}_\rho/\bar{\rho}\bar{\theta} & 0 & 0 \\ 0 & \bar{\rho}/\bar{\theta} & 0 \\ 0 & 0 & \bar{\varepsilon}_\theta\bar{\rho}/\bar{\theta}^2 \end{pmatrix}, \quad A^1 = \begin{pmatrix} \bar{p}_\rho\bar{u}/\bar{\theta}\bar{\rho} & \bar{p}_\rho/\bar{\theta} & 0 \\ \bar{p}_\rho/\bar{\theta} & \bar{u}\bar{\rho}/\bar{\theta} & \bar{p}_\theta/\bar{\theta} \\ 0 & \bar{p}_\theta/\bar{\theta} & \bar{\rho}\bar{\varepsilon}_\theta\bar{u}/\bar{\theta}^2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{\mu}/\bar{\theta} & 0 \\ 0 & 0 & \bar{\alpha}/\bar{\theta}^2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ \bar{\kappa}\bar{\rho}/\bar{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The compensating matrix symbol $K(\xi)$ can be found by inspection with $K(\xi), \xi K(\xi)$ uniformly bounded:

$$K(\xi) = \delta\varepsilon \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \bar{c}\beta(\xi)^{-1/2} \\ 0 & -\bar{c}\beta(\xi)^{-1/2} & 0 \end{pmatrix},$$

$$\bar{c} = \bar{p}_\theta\bar{\theta}^{1/2}/\big(\bar{\varepsilon}_\theta^{1/2}\bar{\rho}\big), \qquad 0 < \delta \ll 1.$$

Global existence and asymptotic decay, non isothermal case

Theorem (P., Valdovinos, 2022)

Assume

$$U_0 := (\rho_0, u_0, \theta_0)^\top \in \left(H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R})\right) \cap \left(L^1(\mathbb{R})\right)^3, \quad s \geq 3.$$

There exists $\delta_2 > 0$ such that if $\|U_0\|_s + \|U_0\|_{L^1} \leq \delta_2$, then the Cauchy problem for the nonlinear system (θK) has a unique global solution $(\rho + \bar{\rho}, u + \bar{u}, \theta + \bar{\theta})(x,t)$ satisfying

$$\begin{aligned}
&\rho \in C\left((0,\infty); H^{s+1}(\mathbb{R})\right) \cap C^1\left((0,\infty); H^{s-1}(\mathbb{R})\right), \\
&u, \theta \in C\left((0,\infty); H^{s}(\mathbb{R})\right) \cap C^1\left((0,\infty); H^{s-2}(\mathbb{R})\right) \\
&(\rho_x, u_x, \theta_x) \in L^2\left((0,\infty); H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\right).
\end{aligned}$$

Furthermore,

$$\|\!|U|\!\|_{s,t} \leq C_2 \|U_0\|_s,$$

$$\|U(t)\|_{s-1} \leq C_1 (1+t)^{-1/4} (\|U_0\|_{s-1} + \|U_0\|_{L^1}),$$

for every $t \in [0, \infty)$.

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$$\rho \in C\left((0,\infty); H^{s+1}(\mathbb{R})\right) \cap C^1\left((0,\infty); H^{s-1}(\mathbb{R})\right),$$
$$u, \theta \in C\left((0,\infty); H^{s}(\mathbb{R})\right) \cap C^1\left((0,\infty); H^{s-2}(\mathbb{R})\right)$$
$$(\rho_x, u_x, \theta_x) \in L^2\left((0,\infty); H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\right).$$

Furthermore,

$$|||U|||_{s,t} \leq C_2 \|U_0\|_s,$$
$$\|U(t)\|_{s-1} \leq C_1 (1+t)^{-1/4} (\|U_0\|_{s-1} + \|U_0\|_{L^1}),$$

for every $t \in [0, \infty)$.

Details in **P, Valdovinos**, Preprint (2022).

Good news: Extension to several space dimensions is feasible. Energy estimates with the same methodology. Applications to:

- quantum hydrodynamics (Bohm potential)
- Korteweg fluids in multi-d
- fourth gradient fluid model (Ruggeri, Gouin)
- "dispersive" Navier-Stokes system (Levermore, Sun)

Details in: **Angeles, P, Valdovinos.** Preprint, 2022.

Obrigado!