UNIVERSIDAD NACIONAL AUTÓNOMA DE MEXICO
PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y
DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

“WELL-POSEDNESS AND DISSIPATIVE STRUCTURE OF THE ONE-DIMENSIONAL SYSTEM FOR COMPRESSIBLE ISOTHERMAL FLUIDS OF KORTEWEG TYPE “

TESINA
QUE PARA OPTAR POR EL GRADO DE:
MAESTRO EN CIENCIAS

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CIUDAD DE MÉXICO, NOVIEMBRE DE 2020.

A mis padres
Leonardo y Rafaela

Agradecimientos

A mis queridos padres y hermanos, por su cariño y apoyo siempre presentes.

Al Dr. Ramón Plaza, mi tutor y director de tesina, por compartirme sus conocimientos y pasión por las Matemáticas.

Al Dr. José de Jesús Álvarez, que lo considero mi padre espiritual, que ha influido profundamente en mi manera de concebir la existencia.

A Víctor y Miguel, mis amigos, por su apoyo y fructuosas pláticas.

A Paulina, por lo mucho que me ofreció y que aprendí de ella.

A la Universidad Nacional Autónoma de México (UNAM).

Al Posgrado en Ciencias Matemáticas de la UNAM.

Al Consejo Nacional de Ciencia y Tecnología, por la beca con número (CVU/Becario) 611532 para mis estudios de maestría.

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1 Introduction

The nature of the interface between two fluids has been the subject of extensive investigations for over two centuries. In the early years of the 1800s, Young, Laplace, and Gauss considered the interface between two fluids to be represented as a surface of zero thickness endowed with physical properties such as surface tension. These investigations, which were based on static and mechanical equilibrium arguments, assumed that physical quantities such as density were, in general, discontinuous across the interface. Physical process such as capillarity occurring in the interface were represented by boundary conditions imposed there (e.g. Young's equation for the equilibrium contact angle or the Young-Laplace equation relating the jump in pressure across the interface to the product of surface tension and curvature). Later, Poisson (1831), Maxwell (1876), and Gibbs (1876) recognized that the interface actually represented a rapid but smooth transition of physical quantities between the bulk fluid values. Gibbs introduced the notion of dividing surface (a *surface of discontinuity*) and surface excess quantities in order to develop the equilibrium thermodynamics of interfaces. The idea that the interface has non-zero thickness (i.e. it is diffusive) was developed in detail by Lord Rayleigh (1892) and by van der Waals (1893), who proposed gradient theories for the interface based on thermodynamic principles. In particular, van der Waals developed a theory of the interface based on his equation of state and used it to predict the thickness of the interface, which he showed became infinite as the critical temperature is approached. Korteweg (1901) built on these ideas and proposed a constitutive law for the capillary stress tensor in terms of the density and its spatial gradients. Specifically, he proposed to study a compressible fluid model in which the "elastic" or "equilibrium" portion of the Cauchy stress tensor $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \left(-p + \alpha\triangle\rho + \beta|\nabla\rho|^2\right)\boldsymbol{I} + \delta\nabla\rho\otimes\nabla\rho + \gamma\nabla\otimes\nabla\rho, \tag{1.1}$$

which can be written in components

$$(\boldsymbol{T})_{ij} = -p\delta_{ij} + \upsilon_{ij},$$

with δ_{ij} being the Kronecker symbol and

$$\upsilon_{ij} = \left(\alpha\triangle\rho + \beta|\nabla\rho|^2\right)\delta_{ij} + \left(\delta\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + \gamma\frac{\partial^2\rho}{\partial x_i\partial x_j}\right),$$

where $\rho = \rho(\boldsymbol{x}, t)$ is the density of the fluid at the point $\boldsymbol{x}$ and time t, $\nabla\rho$ and $\triangle\rho$ are, respectively, the gradient and Laplacian of ρ with respect to $\boldsymbol{x}$ (space), p, α, β, δ, and γ are material functions of ρ and temperature θ, and $(\boldsymbol{a}\otimes\boldsymbol{b})_{ij} = a_i b_j$ is the dyadic product of $\boldsymbol{a}$ and $\boldsymbol{b}$. To include viscous effects into the dynamics of the fluid, Korteweg added to the right-hand side of (1.1) the classical form of the Cauchy and Poisson, i.e., $\lambda(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + 2\mu\boldsymbol{D}$, where

$$\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t\right), \quad \text{here with} \quad (\nabla\boldsymbol{u})_{ij} := \frac{\partial u_j}{\partial x_i},$$

is the usual stress tensor of hydrodynamics, $\boldsymbol{u} = (u_1, u_2, u_3)^T$ being the velocity, and where λ and μ, are the usual viscosity coefficients which may depend on ρ and θ.

Korteweg's form (1.1) is a special example of an elastic material of grade N in which, in order to model more complex spatial interaction effects in a material, the constitutive quantities (here $\boldsymbol{T}$), are allowed to depend not only on the first gradient of the deformation, but also on all gradients of the deformation that are less than or equal to the integer N. However, these high-grade models are, in general, incompatible with the usual continuum theory of thermodynamics. Korteweg model (1.1) is incompatible with conventional thermodynamics unless all the nonclassical coefficients, α, β, δ, and

γ vanish identically. In order to remedy this difficulty, Dunn and Serrin [5] proposed the concept of interstitial working.

Some special cases of Korteweg models arise in quantum mechanics. The motivation for this work is about fluids tough, specially in liquid-vapour mixture with phase changes. The theories of Korteweg's type heve been used intensively to analyse the structure of liquid-vapour phase transition under both static [1] and dynamic [14] conditions. As mentioned, Korteweg model allow phase "boundaries" of nonzero thickness that are often called *diffuse interfaces*, by contrast by the sharp interfaces in the Laplace-Young's theory. In the late 1990's there was a renewal in the interest for diffusive interfaces also for numeric purposes (see [2]).

In the present work, we study the one-dimensional isothermal compressible fluid model of Korteweg type. We consider the special case when the viscosity and capillary coefficients are constants.

In chapter 2 we make a review of the generalization made by Humpherys [10] on the work by Kawashima [13]. In Chapter 3 we study the dissipative structure of our model in the sense of the aforementioned extension. It is worth to note that such study is the first time that is made, or at least reported, since in the literature there are no articles in the subject. We think the generalization made by Humpherys is of relevance, even though the mathematical community has not put much attention on his work, and the present work is a clear example of its applicability.

Chapter 4 deals with the well-posedness of the model in consideration. We note that such study has already been done by Hatorri and Li for the two- and high dimensional case (see [8, 9]), and the proofs presented there are based on the ones presented by the authors in [8], but are independent and adapted to the one-dimensional case. Although the study is made in Eulerian coordinates, contrary to the dissipative structure that is done in Lagrangian, the solution obtained, being classical, is also a solution for the Lagrangian formulation.

Finally, in Chapter 5 we present some results of the linear decay rates of solutions for the linearized system around constant states (Maxwellian). The results presented there exploit the genuinely coupling of the system (which is equivalent to the dissipative structure). The genuine coupling condition, roughly speaking, means that can not exists hyperbolic directions whereby traveling wave solutions type can not be dissipated by the viscous terms.

2 Admissibility of Viscous-Dispersive Systems

Kawasima's theory ([13]) considers the second-order constant coefficient system

$$v_t = Lv := -Av_x + Bv_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad v \in \mathbb{R}^m, \tag{2.1}$$

where A and B are symmetric matrices with B positive semi-definite. Taking the Fourier transform, the evolution of (2.1) reduces to solving the eigenvalue problem

$$\left(\lambda + i\xi A + \xi^2 B\right)\hat{v} = 0. \tag{2.2}$$

We have the theorem:

Theorem 2.1 ***(Shizuta-Kawashima [13]).*** *The following statements are equivalent:*

(i) L is strictly dissipative, that is, $\Re\mathrm{e}\,(\lambda(\xi)) < 0$ for all $\xi \neq 0$.

(ii) L is genuinely coupled, that is, no eigenvalue of A is in $\mathcal{N}(B)$.

(iii) There exists a skew-Hermitian matrix K such that $[K, A] + B > 0$.

Here $\mathcal{N}(B) = \{v \in \mathbb{R}^m \,:\, Bv = 0\}$ and $[K, A] := KA - AK$.

The last theorem was generalized by Humpherys [10] by considering the general linear system

$$v_t = \mathcal{L}v := -\sum_{k=0}^{n} D_k \partial_x^k v, \quad x \in \mathbb{R}, \quad t > 0, \quad v \in \mathbb{R}^m, \tag{2.3}$$

where each $m \times m$ matrix D_k in constant. Likewise, by taking the Fourier transform, the evolution of (2.3) reduces to the eigenvalue problem

$$\lambda \hat{v} + \sum_{k=0}^{n} (i\xi)^k D_k \hat{v} = 0. \tag{2.4}$$

Simplifying by separating the odd- and even- terms in (2.4), we get

$$(\lambda + i\xi A(\xi) + B(\xi))\,\hat{v} = 0, \tag{2.5}$$

where

$$A(\xi) := \sum_{k \text{ odd}} D_k (i\xi)^{k-1} \quad \text{and} \quad B(\xi) := \sum_{k \text{ even}} (-1)^{k/2} D_k \xi^k. \tag{2.6}$$

The matrix $A(\xi)$ and $B(\xi)$ are referred as the generalized flux and the generalized viscosity, respectively. Then we have the following definitions

Definition 2.2

(i) $\mathcal{L}$ is called strictly dissipative if for each $\xi \neq 0$, we have that $\Re e\,(\lambda(\xi)) < 0$.

(ii) $\mathcal{L}$ is said to be genuinely coupled if no eigenvalue of $A(\xi)$ is in $\mathcal{N}(B(\xi))$, for all $\xi \neq 0$.

The main result in [10] is that for symmetric systems, the properties of strict dissipativity, genuine coupling of definition 2.2 , and the existence of a skew-symmetric compensating matrix K are equivalent. The following assumptions are made:

(H1) $A(\xi)$ is symmetric and of constant multiplicity in ξ.

(H2) $B(\xi) \geq 0$ (symmetric and positive semi-definite).

Next, we state the main result without proof and after that we make some important remarks.

Theorem 2.3 *Given (H1) and (H2) above, the following statements are equivalent:*

(i) $\mathcal{L}$ *is strictly dissipative.*

(ii) $\mathcal{L}$ *is genuinely coupled.*

(iii) There exists a real-analytic skew-Hermitian matrix-values $K(\xi)$ *such that* $[K(\xi), A(\xi)] + B(\xi) > 0$ *for all* $\xi \neq 0$.

Remarks:

1. The compensation matrix K is of the form

$$K(\xi) = \sum_{i \neq j} \frac{\pi_{ij}\,(B(\xi))}{\mu_i - \mu_j} = \sum_{i \neq j} \frac{\pi_i B(\xi) \pi_j}{\mu_i - \mu_j}, \tag{2.7}$$

   that is the Drazin inverse or reduced resolvent of the commutator operator, where $\{\mu_j\}_{j=1}^r$ denote the distinct eigenvalues of $A(\xi)$ with corresponding eigenprojections $\{\pi_j\}_{j=1}^r$

2. Theorem 2.3 can be extended in the next setting: First we have the following definition

Definition 2.4 *$\mathcal{L}$ is called symmetrizable in the sense of Humpherys [10] if there exists a symmetric, real-analytic matrix-valued $A_0(\xi) > 0$ such that both $A_0(\xi)A(\xi)$ and $A_0(\xi)B(\xi)$ are symmetric, and $A_0(\xi)B(\xi) \geq 0$. We say that $A_0(\xi)$ is a symmetrizer of $\mathcal{L}$.*

We notice that this notion of symmetrizability differs from the Friedrichs' one [7]. The last is a symmetrization term-wise. Actually our model in not symmetrizable in the sense of Friedrich, but it is in the sense of Humpherys (see [10]).
With this more general notion of symmetrizability, we can extend easily Theorem 2.3 to the following:

Theorem 2.5 *(**Symmetrizable version**).If $A_0(\xi)$ is a symmetrizer of $\mathcal{L}$, then the following statements are equivalent:*

(i) $\mathcal{L}$ is strictly dissipative.

(ii) $\mathcal{L}$ is genuinely coupled.

(iii) There exists a real-analytic skew-Hermitian matrix-valued $K(\xi)$ such that $[K(\xi), A_0(\xi)A(\xi)] + A_0(\xi)B(\xi) > 0$ for all $\xi \neq 0$.

It is the last generalization of the standard definition of symmetrizability the one that we adopt here.

3 Genuine coupling of the one-dimensional Korteweg model

3.1 The Korteweg model

Models of Korteweg-type are obtained from an extended version of nonequilibrium thermodynamics, in which it is assumed that the energy of the fluid not only depends on standard variables but on the gradient of the density ([4]). In terms of the free energy, this principle takes the form of the generalized Gibbs relation

$$\mathrm{d}F = -S\mathrm{d}T + g\mathrm{d}\rho + \phi \cdot \mathrm{d}\boldsymbol{w},$$

where F denotes the free energy per unit volume, S the entropy per unit volume, T the temperature, g the chemical potential and, in the additional term, $\boldsymbol{w}$ stands for $\nabla\rho$. The potential ϕ is often assumed, and it will be in our case, to be of the form

$$\phi = K\boldsymbol{w},$$

where K is called the capillarity coefficient, which may depend on ρ and T. In this case, F can be decomposed into a standard part and an additional term due to gradient of density,

$$F(\rho, T, \nabla\rho) = F_0(\rho, T) = +\frac{1}{2}K(\rho, T)|\nabla\rho|^2,$$

and similar decompositions hold for S and g. Then, we define the Korteweg tensor as

$$\boldsymbol{K} := (\rho\,\mathrm{div}\phi)\,\boldsymbol{I} - \phi^T\boldsymbol{w}.$$

Neglecting dissipation phenomena, the conservation of mass, momentum and energy read

$$\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u}) = 0,$$

$$\partial_t(\rho\boldsymbol{u}) + \mathrm{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla\tilde{p} = \mathrm{div}\boldsymbol{K},$$

$$\partial_t\left(E + \frac{1}{2}\rho|\boldsymbol{u}|^2\right) + \mathrm{div}\left(\left(E + \frac{1}{2}\rho|\boldsymbol{u}|^2 + \tilde{p}\right)\boldsymbol{u}\right) = \mathrm{div}(\boldsymbol{K}\boldsymbol{u} + \boldsymbol{W}),$$

where $\tilde{p} = \rho g - F$ is the (extended) pressure, $E = F + TS$ is the internal energy per unit volume, and

$$\boldsymbol{W} := \left(\partial_t \rho + \boldsymbol{u}^T \cdot \nabla \rho\right) \phi^T = -\left(\rho \operatorname{div} \boldsymbol{u}\right) \phi^T,$$

is the interstitial work that was first introduced by Dunn and Serrin [5]. This additional term ensures that the entropy S satisfies the conservation law

$$\partial_t S + \operatorname{div}(S\boldsymbol{u}) = 0.$$

(This is obtained by formal computation, assuming smooth solutions). As we mentioned, we assume that $\phi = K\boldsymbol{w}$. Then we can write

$$g = g_0 + \frac{1}{2} K'_\rho |\nabla \rho|^2,$$

where g_0 is independent of $\nabla \rho$.

We are interested in studying the isothermal Korteweg model with viscosity, in one-dimensional space. Let us write the system we are going to deal with throughout the rest of the work. Noting that

$$\tilde{p} = p + \frac{1}{2}\left(\rho K'_\rho - K\right)|\nabla \rho|^2, \quad p = \rho g_0 - F_0,$$

our system can be written as

$$\begin{aligned} &\partial_t \rho + \partial_x (\rho u) = 0, \\ &\partial_t (\rho u) + \partial_x \left(\rho u^2 + p - \rho K \rho_{xx} + \tfrac{1}{2} K \rho_x^2 - \tfrac{1}{2} \rho K'_\rho \rho_x^2 - \mu u_x\right) = 0. \end{aligned} \tag{3.1}$$

3.2 Fluid dynamics in Lagrangian variables

Let us consider the conservation law of mass

$$\rho_t + (\rho u)_x = 0 \tag{3.2}$$

which makes no appeal to any approximation. It expresses that the differential form $\alpha := \rho \mathrm{d}x - \rho u \mathrm{d}t$ is closed an therefore exact (the domain needs to be simply connected). We thus introduce a function $(x,t) \longrightarrow y$, defined to within a constant by $\alpha = \mathrm{d}y$ (that is, y is the function whose derivative is the $1-$ form α). We have $\mathrm{d}x = u\mathrm{d}t + \tau \mathrm{d}y$, where $\tau = \rho^{-1}$ is the *specific volume*. Given another conservation law $\partial_t u_i + \partial_x q_i = 0$, which can be written $\mathrm{d}(q_i \mathrm{d}t - u_i \mathrm{d}x) = 0$, we have

$$\begin{aligned} \mathrm{d}\left((q_i - u_i u)\,\mathrm{d}t - u_i \tau \mathrm{d}y\right) &= \mathrm{d}\left((q_i - u_i u)\,\mathrm{d}t - u_i \tau\left(\rho \mathrm{d}x - \rho u \mathrm{d}t\right)\right) \\ &= \mathrm{d}\left((q_i - u_i u)\,\mathrm{d}t - u_i \mathrm{d}x + u_i u \mathrm{d}t\right) \\ &= d\left(q_i \mathrm{d}t - u_i \mathrm{d}x\right) = 0. \end{aligned}$$

That is

$$\partial_t (u_i \tau) + \partial_y (q_i - u_i u) = 0. \tag{3.3}$$

The system written in the variables (y,t) is thus formed of conservation laws.
Let us look at some examples. In the absence of viscosity, we have the momentum conservation equation

$$\partial_t (\rho u) + \left(\rho u^2 + p(\rho, e)\right)_x = 0.$$

Then, taking $u_2 = \rho u$ and $q_2 = \rho u^2 + p(\rho, e)$ in (3.3), we obtain

$$\partial_t u + \partial_y P(\tau, e) = 0, \tag{3.4}$$

where $P(\tau, e) := p(\tau^{-1}, e)$. Similarly, for the energy conservation equation

$$\partial_t E + ((E+p)\,u)_x = 0,$$

where $E = \frac{1}{2}\rho v^2 + \rho e$, taking $u_3 = E$ and $q_3 = (u_3 + p)\,v$ in (3.3), we obtain:

$$\partial_t \left(\frac{1}{2} u^2 + e \right) + \partial_y \left(P\left(\tau, e\right) u \right) = 0. \tag{3.5}$$

The conservation of mass gives nothing new since it was already used to construct the change of variables (actually one gets the trivial equation $1_t + 0_y = 0$). To complete the system of equations for the unknowns (τ, u, e) we have to include a trivial conservation law. For example, if we take $u_4 \equiv 1$ y $q \equiv 0$, we get

$$\partial_t \tau = \partial_y u. \tag{3.6}$$

Remark: Since we are defining y to be such that $\mathrm{d}y = \alpha = \rho \mathrm{d}x - \rho u \mathrm{d}t$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \rho.$$

Thus giving any function $v = v(x,t)$, by the chain rule

$$v_x = v_y \frac{\mathrm{d}y}{\mathrm{d}x} = v_y \rho = v_y \tau^{-1},$$

we obtain

$$v_y = \tau v_x.$$

3.3 Viscous-capillary model in Lagrangian variables

Now, let us write our model, using the material of the previous section, in lagrangian coordinates.

The system in question is the conservation law of mass (3.2) together with the balance of momentum equation

$$(\rho u)_t + \left(\rho u^2 + p - \rho K \rho_{xx} + \frac{1}{2} K \rho_x^2 - \frac{1}{2} \rho K'_\rho \rho_x^2 - \mu u_x \right)_x = 0, \tag{3.7}$$

where K'_ρ is the derivative of K with respect to ρ. In this case $u_i = \rho v$ and $q_i = \rho v^2 + p - \rho K \rho_{xx} + \frac{1}{2} K \rho_x^2 - \frac{1}{2} \rho K' \rho_x^2 - \mu v_x$, so that we obtain

$$u_t + \partial_y \left(p - \rho K \rho_{xx} + \frac{1}{2} K \rho_x^2 - \frac{1}{2} \rho K' \rho_x^2 - \mu u_x \right) = 0.$$

As we can see, the last expression contains derivatives with respect to x. To write the expression with derivatives with respect to y only, we make use of the remark of the previous section. First, since $\tau \rho_x = \rho_y$, we have

$$\rho_x = \frac{1}{\tau} \left(\frac{1}{\tau} \right)_y = -\frac{\tau_y}{\tau^3}.$$

Then

$$\rho_{xx} = -\frac{1}{\tau} \partial_y \left(\frac{\tau_y}{\tau^3} \right) = -\frac{1}{\tau} \left(\frac{\tau_{yy}}{\tau^3} - \frac{3\tau_y^2}{\tau^4} \right).$$

Also we have

$$K'_\rho = K'_\tau \frac{\mathrm{d}\tau}{\mathrm{d}\rho} = K'_\tau \left(\frac{1}{\rho}\right)_\rho = -\frac{K'_\tau}{\rho^2} = -\tau^2 K'_\tau,$$

so that

$$\begin{aligned}
-\rho K \rho_{xx} + \frac{1}{2} K \rho_x^2 - \frac{1}{2}\rho K' \rho_x^2 &= K\left(\frac{\tau_{yy}}{\tau^5} - \frac{3\tau_y^2}{\tau^6}\right) + \frac{1}{2} K \frac{\tau_y^2}{\tau^6} + \frac{1}{2}\tau K'_\tau \frac{\tau_y^2}{\tau^6} \\
&= \frac{K}{\tau^5}\tau_{yy} - \frac{5}{2} K \frac{\tau_y^2}{\tau^6} + \frac{1}{2} K'_\tau \frac{\tau_y^2}{\tau^5} \\
&= k\tau_{yy} + \frac{1}{2} k'_\tau \left(\tau_y\right)^2,
\end{aligned}$$

where $k(\tau) = K(1/\tau)\left(1/\tau^5\right)$. Thus, together with the conservation law (3.6), we obtain the next system of equations:

$$\partial_t \tau - \partial_y u = 0, \tag{3.8a}$$

$$\partial_t u + \partial_y p = -\partial_y \left(k \partial_y^2 \tau + \frac{1}{2} k'_\tau \left(\partial_y \tau\right)^2 - \frac{\mu}{\tau} u_y\right). \tag{3.8b}$$

Since we are considering the isothermal case, we have that p is a function of ρ only, that is

$$p = p(\rho) = p(\tau),$$

and assume that $p'(\tau) < 0$, where the derivative is taken with respect to τ, for all τ. Then, we get

$$\partial_y p = \frac{\mathrm{d}p}{\mathrm{d}\tau} \tau_y.$$

In addition,

$$\begin{aligned}
\partial_y \left(k \partial_y^2 \tau\right) &= k'_\tau \partial_y \tau \partial_y^2 \tau + k \partial_y^3 \tau, \\
\partial_y \left(\frac{1}{2} k'_\tau \left(\partial_y \tau\right)^2\right) &= \frac{1}{2} k''_\tau \left(\partial_y \tau\right)^3 + k'_\tau \partial_y \tau \partial_y^2 \tau, \\
\partial_y \left(\frac{\mu}{\tau} u_y\right) &= \frac{\partial_y \mu}{\tau} \partial_y u + \frac{\mu}{\tau} \partial_y^2 u - \frac{\mu \partial_y \tau}{\tau^2} \partial_y u.
\end{aligned}$$

Then the system (3.8) can be written as

$$\partial_t \tau - \partial_y u = 0, \tag{3.9a}$$

$$\begin{aligned}
\partial_t u + p'(\tau) \partial_y \tau = -\Bigg(& k \partial_y^3 \tau - \frac{\mu}{\tau} \partial_y^2 u + 2 k'_\tau \partial_y \tau \partial_y^2 \tau + \frac{1}{2} k''_\tau \left(\partial_y \tau\right)^3 \\
& - \left\{\frac{\partial_y \mu}{\tau} \partial_y u - \frac{\mu \partial_y \tau}{\tau^2} \partial_y u\right\}\Bigg).
\end{aligned} \tag{3.9b}$$

If we define $U = \begin{pmatrix} \tau \\ u \end{pmatrix}$, we can write the above system in the form of (2.3)

$$U_t = -D_1 U_y - D_2 U_{yy} - D_3 U_{yyy} - G\left(U, U_y, U_{yy}\right) \tag{3.10}$$

where

$$D_1 = \begin{pmatrix} 0 & -1 \\ p'(\tau) & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\mu}{\tau} \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix},$$

and

$$G\left(U, U_y, U_{yy}\right) = \begin{pmatrix} 0 \\ 2k'_\tau \partial_y \tau \partial_y^2 \tau + \frac{1}{2} k''_\tau \left(\partial_y \tau\right)^3 - \left\{\frac{\partial_y \mu}{\tau} \partial_y u - \frac{\mu \partial_y \tau}{\tau^2} \partial_y u\right\} \end{pmatrix}.$$

3.4 Genuine coupling of the Korteweg model

In order to define about the property of genuine coupling for our system of interest, let us consider a constant equilibrium state $\bar{U} = (\bar{\tau}, \bar{u})$ for the specific volume and velocity field. If $\bar{U} + U$ is a solution of (3.10), we can rewrite the system as

$$U_t + D_1(\bar{U})U_y + D_2(\bar{U})U_{yy} + D_3(\bar{U})U_{yyy} = \mathcal{N}\left(U, U_y, U_{yy}, U_{yyy}\right), \tag{3.11}$$

where $\mathcal{N}$ contains the non-linear terms. Let us consider the linear part of (3.11), that is, the linear system

$$U_t + D_1(\bar{U})U_y + D_2(\bar{U})U_{yy} + D_3(\bar{U})U_{yyy} = 0. \tag{3.12}$$

Then, according to (2.6), we have

$$B\left(\xi\right) = -\xi^2 D_2(\bar{U}) = \xi^2 \begin{pmatrix} 0 & 0 \\ 0 & \frac{\bar{\mu}}{\bar{\tau}} \end{pmatrix},$$

and

$$A\left(\xi\right) = D_1(\bar{U}) - \xi^2 D_3(\bar{U}) = \begin{pmatrix} 0 & -1 \\ p'(\bar{\tau}) & 0 \end{pmatrix} - \xi^2 \begin{pmatrix} 0 & 0 \\ \bar{k} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ p'(\bar{\tau}) - \xi^2\bar{k} & 0 \end{pmatrix},$$

where $\bar{\mu} = \mu(\bar{u})$ and $\bar{k} = k(\bar{\tau})$.

Since $A(\xi)$ is not symmetric, we would like to symmetrize the system, that is, to find a symmetric matrix, real analytic $A_0\left(\xi\right) > 0$ such that $A_0\left(\xi\right)A\left(\xi\right)$ and $A_0\left(\xi\right)B\left(\xi\right)$ are symmetric, and $A_0\left(\xi\right)B\left(\xi\right) \geq 0$. Let us take

$$A_0\left(\xi\right) = \begin{pmatrix} \xi^2\bar{k} - p'(\bar{\tau}) & 0 \\ 0 & 1 \end{pmatrix},$$

which is symmetric, real analytic and $A_0\left(\xi\right) > 0$. Also, we have

$$\hat{A}\left(\xi\right) = A_0\left(\xi\right)A\left(\xi\right) = \begin{pmatrix} 0 & p'(\bar{\tau}) - \xi^2\bar{k} \\ p'(\bar{\tau}) - \xi^2\bar{k} & 0 \end{pmatrix},$$

and

$$\hat{B}\left(\xi\right) = A_0\left(\xi\right)B\left(\xi\right) = \xi^2 \begin{pmatrix} 0 & 0 \\ 0 & \frac{\bar{\mu}}{\bar{\tau}} \end{pmatrix},$$

with $A_0\left(\xi\right)A\left(\xi\right)$ and $A_0\left(\xi\right)B\left(\xi\right)$ symmetric and $A_0\left(\xi\right)B\left(\xi\right) \geq 0$.

Let us see that the system in question satisfies the property of genuinely coupling. To see that, we have to verify that no eigenvector of $\hat{A}\left(\xi\right)$ is in the null space of $\hat{B}\left(\xi\right)$. By the form of $\hat{A}$, it is easy to see that its eigenvalues are

$$\lambda_{1,2} = \pm\beta$$

where $\beta(\xi) = \xi^2\bar{k} + q(\bar{\tau})$ and $\bar{q} = -p'(\bar{\tau})$, with eigenvector $v_1 = (1, -1)^T$ and $v_2 = (1, 1)^T$, respectively. Then

$$\hat{B}\left(\xi\right)v_1 = \xi^2 \begin{pmatrix} 0 \\ -\frac{\bar{\mu}}{\bar{\tau}} \end{pmatrix} \neq \mathbf{0}, \qquad \hat{B}\left(\xi\right)v_2 = \xi^2 \begin{pmatrix} 0 \\ \frac{\bar{\mu}}{\bar{\tau}} \end{pmatrix} \neq \mathbf{0},$$

which means that the system us genuinely coupled.

In what follows we are going to compute the compensation matrix. First, let us observe that

$$(1,0)^T = \frac{1}{2}\left(1,-1\right)^T + \frac{1}{2}\left(1,1\right)^T,$$

$$(0,1)^T = -\frac{1}{2}\left(1,-1\right)^T + \frac{1}{2}\left(1,1\right)^T,$$

so that the matrix representation of the projection $\pi_1(\xi)$ y $\pi_2(\xi)$ under the subspaces generated by v_1 and v_1 are

$$\pi_1(\xi) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \text{ and } \quad \pi_2(\xi) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

respectively. Then, after making the computations, we get

$$\frac{\pi_1 \hat{B} \pi_2}{\lambda_1 - \lambda_2}(\xi) = \frac{\xi^2 \bar{\mu}}{8\beta(\xi)\bar{\tau}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$$

$$\frac{\pi_2 \hat{B} \pi_1}{\lambda_2 - \lambda_1}(\xi) = \frac{\xi^2 \bar{\mu}}{8\beta(\xi)\bar{\tau}}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Then according to (2.7),

$$K(\xi) = \frac{\pi_1 \hat{B} \pi_2}{\lambda_1 - \lambda_2}(\xi) + \frac{\pi_2 \hat{B} \pi_1}{\lambda_2 - \lambda_1}(\xi) = \frac{\xi^2 \bar{\mu}}{4\beta(\xi)\bar{\tau}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.13}$$

Clearly, $K(\xi)$ skew-Hermitian, and

$$\begin{aligned} \left[K(\xi), \hat{A}(\xi)\right] &= K(\xi)\hat{A}(\xi) - \hat{A}(\xi)K(\xi) \\ &= \frac{\xi^2 \bar{\mu}}{4\beta(\xi)\bar{\tau}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\beta(\xi) \\ -\beta(\xi) & 0 \end{pmatrix} \\ &\quad - \frac{\xi^2 \bar{\mu}}{4\beta(\xi)\bar{\tau}}\begin{pmatrix} 0 & -\beta(\xi) \\ -\beta(\xi) & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{\xi^2 \bar{\mu}}{4\bar{\tau}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\xi^2 \bar{\mu}}{4\bar{\tau}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\xi^2 \bar{\mu}}{2\bar{\tau}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned}$$

so that

$$\begin{aligned} \left[K(\xi), \hat{A}(\xi)\right] + \hat{B}(\xi) &= \frac{\xi^2 \bar{\mu}}{2\bar{\tau}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\xi^2 \bar{\mu}}{\bar{\tau}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{\xi^2 \bar{\mu}}{2\bar{\tau}}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\right] \\ &= \frac{\xi^2 \bar{\mu}}{2\bar{\tau}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

which is a definite positive matrix. Thus the matrix $K(\xi)$ is the required compensating matrix. It is worth to note that the matrix $K(\xi)$ is compensating function in the sense of Humpherys.

4 Well-Posedness of the one-dimensional model for material of Korteweg Type

In this section we are going to show the short-time existence of solution to our model. All the treatment is going to be done for the model in Eulerian coordinates. Since the solution obtained is classical, by a comment in [15], it is also a solution for the model in Lagrangian coordinates.
Let us recast the model as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2\right)_x = \left\{-p - \frac{\nu}{2}\rho_x^2 + \nu\rho\rho_{xx} + \frac{4}{3}\mu u_x\right\}_x, \end{cases} \tag{4.1}$$

where we have taken $K = \nu$ and μ equal to a constant $(4/3)\mu$ in (3.1), which can be written in quasilinear form

$$\mathcal{L}(w)\, w \equiv \begin{cases} (\partial_t + u\partial_x)\,\rho + \rho u_x = 0, \\ (\partial_t + u\partial_x)\, u + p'(\rho)\,\rho^{-1}\partial_x\rho - \nu\triangle\partial_x\rho - \frac{4}{3}\mu\rho^{-1}\triangle u = 0. \end{cases} \tag{4.2}$$

We consider the Cauchy problem for (4.2) with initial condition

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x). \tag{4.3}$$

For this purpose, let us first consider the linearized version for the perturbation $(\tilde{\rho}, \tilde{u})$ at a given solution (ρ, u) of problem (4.2) and (4.3), that is,

$$\begin{cases} (\partial_t + u\partial_x)\,\tilde{\rho} + \rho\tilde{u}_x = \tilde{f}_1, \\ \\ (\partial_t + u\partial_x)\,\tilde{u} + p'(\rho)\,\rho^{-1}\tilde{\rho}_x - \nu\tilde{\rho}_{xxx} - \frac{4}{3}\mu\rho^{-1}\tilde{u}_{xx} = \tilde{f}_2. \end{cases} \tag{4.4}$$

The reason why we study this problem is going to become clear in Section §4.3.

4.1 Linearized problem and a priori energy estimates

Let us consider the Cauchy problem of (4.4) with initial data:

$$\tilde{\rho}(x, 0) = \tilde{\rho}_0(x), \quad \tilde{u}(x, 0) = \tilde{u}_0(x). \tag{4.5}$$

Equation (4.4) can be written in matrix form:

$$\tilde{L}(w)\,\tilde{w} \equiv \begin{pmatrix} \tilde{L}_1(w)\,\tilde{w} \\ \tilde{L}_2(w)\,\tilde{w} \end{pmatrix} \equiv \partial_t\tilde{w} + A_1\partial_x\tilde{w} + (T_1 + T_2)\,\tilde{w} = \tilde{f}, \tag{4.6}$$

where A_1 is the coefficient matrix of the first-order space derivative terms and $T_1 + T_2$ is an operator involving derivatives of order two or higher:

$$A_1 = \begin{pmatrix} u & \rho \\ p'(\rho)\,\rho^{-1} & u \end{pmatrix}, \tag{4.7}$$

$$T_1\tilde{w} = -\nu\begin{pmatrix} 0 \\ \triangle\tilde{\rho}_x \end{pmatrix}, \quad T_2\tilde{w} = -\mu\rho^{-1}\begin{pmatrix} 0 \\ \frac{4}{3}\tilde{u}_{xx} \end{pmatrix}. \tag{4.8}$$

Let $\langle\cdot,\cdot\rangle$ denote the L^2 inner product in $x \in \mathbb{R}$. Denote $\|\cdot\| \equiv \|\cdot\|_0$, the corresponding norm, and $\|\cdot\|_k$, the kth order Sobolev norm.

Let β_0 be a constant such that the variables $w = (\rho, u)$ in the coefficients of (4.4) that satisfy

$$\sup_{x,t}\left(\rho^{-1} + |\rho_t| + \sum_{|j|\le 2} |D_x^j w|\right) \le \beta_0.$$

Let C_0 denote the constant that depends only upon β_0. Then we have the following zero-order energy estimate.

Theorem 4.1 *The smooth solution $\tilde{w} \in C_0^\infty([0,T]\times\mathbb{R})$ of (4.4) and (4.5) satisfies the estimate*

$$\partial_t\left(\|\tilde{w}\|^2 + \|\tilde{\rho}_x\|^2\right) + \|\tilde{u}_x\|^2 \le C_0\left(\|\tilde{w}\|^2 + \|\tilde{\rho}_x\|^2 + \|\tilde{f}\|^2 + \|\tilde{f}_1\|_1^2\right) \tag{4.9}$$

and

$$\|\tilde{w}\|^2_{[0,T]} \le C_0(T)\left(\|\tilde{w}_0\|^2 + \|\tilde{\rho}_0\|_1^2 + \int_0^T \left(\|\tilde{f}(t)\|^2 + \|\tilde{f}_1(t)\|_1^2\right) dt\right). \tag{4.10}$$

Here the norm $\|\tilde{w}\|^2_{[0,T]}$ is defined as follows:

$$\|\tilde{w}\|^2_{[0,T]} = \sup_{0\le t\le T}\left(\|\tilde{w}(t)\|^2 + \|\tilde{\rho}_x(t)\|^2\right) + \int_0^T \|\tilde{u}_x(t)\|^2 dt. \tag{4.11}$$

Proof. Let us define the diagonal matrix

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}.$$

Taking the L^2 inner product of (4.6) with the vector $A_0\tilde{w} = (\tilde{\rho}, \rho\tilde{u})$, we obtain

$$\begin{aligned} \langle \partial_t \tilde{w}, A_0\tilde{w}\rangle + \langle (T_1+T_2)\,\tilde{w}, A_0\tilde{w}\rangle &= -\langle A_1\partial_x\tilde{w}, A_0\tilde{w}\rangle + \langle \tilde{f}, A_0\tilde{w}\rangle \\ &\le \|\tilde{f}\|\|A_0\tilde{w}\| - \langle A_1\partial_x\tilde{w}, A_0\tilde{w}\rangle \\ &\le C_0\left(\|\tilde{f}\| + \|\tilde{w}\| + \|\tilde{\rho}_x\|\right)\|\tilde{w}\| \\ &\le C_0\left(\|\tilde{w}\|^2 + \|\tilde{\rho}_x\|^2 + \|\tilde{f}\|^2\right), \end{aligned} \tag{4.12}$$

where we have used that

$$\begin{aligned} \langle A_1\partial_x\tilde{w}, A_0\tilde{w}\rangle &= \langle \left(u\tilde{\rho}_x + \rho\tilde{u}_x, p'(\rho)\rho^{-1}\tilde{\rho}_x + u\tilde{u}_x\right), (\tilde{\rho}, \rho\tilde{u})\rangle \\ &= \int_{\mathbb{R}} \{u\tilde{\rho}_x\tilde{\rho} + \rho\tilde{\rho}\tilde{u}_x + p'(\rho)\tilde{u}\tilde{\rho}_x + u\rho\tilde{u}_x\tilde{u}\}\,dx \\ &= \int_{\mathbb{R}} \left\{\frac{1}{2}u\partial_x(\tilde{\rho})^2 - \tilde{u}\partial_x(\rho\tilde{\rho}) + p'(\rho)\tilde{u}\tilde{\rho}_x + \frac{1}{2}u\rho\partial_x(\tilde{u})^2\right\} dx \\ &\le -\frac{1}{2}\int_{\mathbb{R}} u_x\tilde{\rho}^2 dx + C_0\|\tilde{u}\|\|\tilde{\rho}_x\| - \frac{1}{2}\int_{\mathbb{R}} \partial_x(u\rho)\,\tilde{u}^2 dx \\ &\le C_0\|\tilde{u}\|\|\tilde{\rho}_x\| + C_0\|\tilde{w}\|^2. \end{aligned}$$

Now, since

$$\frac{1}{2}\partial_t\|\sqrt{A_0}\tilde{w}\|^2 = \frac{1}{2}\partial_t\int_{\mathbb{R}}\left(\tilde{\rho}^2 + \rho\tilde{u}^2\right)dx = \int_{\mathbb{R}}\left(\tilde{\rho}_t\tilde{\rho} + \rho\tilde{u}_t\tilde{u} + \tilde{u}^2\rho_t\right)dx = \langle \partial_t\tilde{w}, A_0\tilde{w}\rangle + \int_{\mathbb{R}} \tilde{u}^2\rho_t dx,$$

we have

$$\langle \partial_t\tilde{w}, A_0\tilde{w}\rangle \ge \frac{1}{2}\partial_t\|\sqrt{A_0}\tilde{w}\|^2 - C_0\|\tilde{w}\|^2. \tag{4.13}$$

From the first equation in (4.4),

$$\rho\tilde{u}_x = -\left(\partial_t + u\partial_x\right)\tilde{\rho} + \tilde{f}_1,$$

and by standard integration by parts we obtain

$$
\begin{aligned}
\langle T_1\tilde{w}, A_0\tilde{w}\rangle &= \langle (0, -\nu\triangle\tilde{\rho}_x), (\tilde{\rho}, \rho\tilde{u})\rangle \\
&= -\nu\int_{\mathbb{R}} (\triangle\tilde{\rho}_x)(\rho\tilde{u})\,dx \\
&= \nu\int_{\mathbb{R}} (\triangle\tilde{\rho})(\rho\tilde{u})_x\,dx \\
&= \langle \nu\triangle\tilde{\rho}, \rho\tilde{u}_x + \rho_x\tilde{u}\rangle \\
&= \langle \nu\triangle\tilde{\rho}, -(\partial_t + u\partial_x)\tilde{\rho} + \tilde{f}_1 + \rho_x\tilde{u}\rangle \\
&= \nu\langle\tilde{\rho}_x, \partial_t\tilde{\rho}_x\rangle - \nu\int_{\mathbb{R}}\tilde{\rho}_{xx}u\tilde{\rho}_x dx + \nu\int_{\mathbb{R}}\tilde{\rho}_{xx}\tilde{f}_1 dx + \nu\int_{\mathbb{R}}\tilde{\rho}_{xx}\rho_x\tilde{u}dx \qquad (4.14)\\
&= \nu\left(\frac{1}{2}\partial_t\|\tilde{\rho}_x\|^2 + \frac{1}{2}\int_{\mathbb{R}} u\partial_x(\tilde{\rho}_x)^2\,dx - \int_{\mathbb{R}}\tilde{\rho}_x\partial_x\tilde{f}_1 dx - \nu\int_{\mathbb{R}}\tilde{\rho}_x\partial_x(\rho_x u)dx\right) \\
&= \nu\left(\frac{1}{2}\partial_t\|\tilde{\rho}_x\|^2 + \frac{1}{2}\int_{\mathbb{R}} u\partial_x(\tilde{\rho}_x)^2\,dx - \int_{\mathbb{R}}\tilde{\rho}_x\partial_x\tilde{f}_1 dx - \nu\int_{\mathbb{R}}\tilde{\rho}_x(\rho_{xx}\tilde{u} + \rho_x\tilde{u}_x)dx\right) \\
&\geq \frac{\nu}{2}\partial_t\|\tilde{\rho}_x\|^2 - C_0\left(\|\tilde{\rho}_x\|^2 + \|\tilde{\rho}_x\|\|\tilde{u}\| + \|\tilde{\rho}_x\|\|\tilde{u}_x\| + \|\tilde{f}_1\|_1\|\tilde{\rho}_x\|\right) \\
&\geq \frac{\nu}{2}\partial_t\|\tilde{\rho}_x\|^2 - C_0\left(\|\tilde{\rho}_x\|^2 + \|\tilde{w}\|^2 + \|\tilde{f}_1\|_1^2\right) - \epsilon\|\tilde{u}_x\|^2,
\end{aligned}
$$

where we used

$$\langle \nu\tilde{\rho}_{xx}, -\partial_t\tilde{\rho}\rangle = \nu\langle\tilde{\rho}_x, \partial_t\tilde{\rho}_x\rangle = \frac{\nu}{2}\partial_t\|\tilde{\rho}_x\|^2.$$

For the term involving T_2 we have

$$
\begin{aligned}
\langle T_2\tilde{w}, A_0\tilde{w}\rangle &= \langle\left(0, -\frac{4}{3}\mu\rho^{-1}\tilde{u}_{xx}\right), (\tilde{\rho}, \rho\tilde{u})\rangle \\
&= -\frac{4}{3}\int_{\mathbb{R}}\mu\tilde{u}_{xx}\tilde{u}_x dx = \frac{4}{3}\mu\int_{\mathbb{R}}\tilde{u}_x^2 dx \qquad (4.15)\\
&= \frac{4}{3}\mu\|\tilde{u}_x\|^2 \geq \mu\|\tilde{u}_x\|^2.
\end{aligned}
$$

Then, combining (4.12)-(4.15) and noticing that $\|A_0\tilde{w}\| \sim \|\tilde{w}\|$ we obtain the estimate (4.9).

From (4.9), using the Gronwall's lemma, we obtain

$$\|\tilde{w}(t)\|^2 + \|\tilde{\rho}_x(t)\|^2 \leq e^{C_0 t}\left(\|\tilde{w}_0\|^2 + \|\tilde{\rho}_0\|^2\right) + \int_0^t e^{C_0(t-s)}\left(\|\tilde{f}(s)\|^2 + \|\tilde{f}_1(s)\|_1^2\right)ds. \qquad (4.16)$$

Replacing the term $\|\tilde{w}\|^2 + \|\tilde{\rho}_x\|^2$ on the right of (4.9), we get

$$
\begin{aligned}
\partial_t\left(\|\tilde{w}\|^2 + \|\tilde{\rho}_x\|^2\right) + \|\tilde{u}_x\|^2 &\leq C_0\left(e^{C_0 t}\left(\|\tilde{w}_0\|^2 + \|\tilde{\rho}_{0x}\|^2\right) + e^{C_0 t}\int_0^t e^{-C_0 s}\left(\|\tilde{f}(s)\|^2 + \|\tilde{f}_1(s)\|_1^2\right)ds\right) \\
&\quad + C_0\left(\|\tilde{f}(t)\|^2 + \|\tilde{f}_1(t)\|_1^2\right) \\
&\leq C_0\left(e^{C_0 t}\left(\|\tilde{w}_0\|^2 + \|\tilde{\rho}_{0x}\|^2\right) + e^{C_0 t}\int_0^T\left(\|\tilde{f}(s)\|^2 + \|\tilde{f}_1(s)\|_1^2\right)ds\right) \\
&\quad + C_0\left(\|\tilde{f}(t)\|^2 + \|\tilde{f}_1(t)\|_1^2\right).
\end{aligned}
$$

Finally, integrating the last expression from 0 to t_1, we obtain

$$\int_0^{t_1} \partial_t \left(\|\tilde{w}\|^2 + \|\tilde{\rho}_x\|^2\right) dt + \int_0^{t_1} \|\tilde{u}_x(t)\|^2 dt \leq \left(\int_0^{t_1} C_0 e^{C_0 t} dt\right) \left(\|\tilde{w}_0\|^2 + \|\tilde{\rho}_{0x}\|^2\right)$$
$$+ C_0 \int_0^{t_1} \left(\|\tilde{f}(t)\|^2 + \|\tilde{f}_1(t)\|_1^2\right) dt$$
$$+ \left(\int_0^{t_1} e^{C_0 t} dt\right) \int_0^T \left(\|\tilde{f}(s)\|^2 + \|\tilde{f}_1(s)\|_1^2\right) ds,$$

yielding (4.10).■

Now we derive high-order estimates. We denote

$$\|\tilde{w}\|^2_{k,[0,T]} \equiv \sum_{j\leq k} \|D_x^j \tilde{w}\|^2_{[0,T]}.$$

Let β_k be a constant such that the variable $w = (\rho, u)$ in the coefficients of (4.6) satisfies

$$\sup_{t,x} \left(\rho^{-1} + |\rho_t| + \sum_{j\leq 2} |D_x^k w|\right) + \|w\|^2_{k,[0,T]} \leq \beta_k. \tag{4.17}$$

Let C_k denote the constant that depends only upon β_k. Then we have the following kth order energy estimate.

Theorem 4.2 *For any integer $k \geq 4$, the smooth solution $\tilde{w} \in C_0^\infty([0,T] \times \mathbb{R})$ of (4.4) and (4.5) satisfies the estimates*

$$\begin{aligned} &\partial_t \left(\|\tilde{w}\|_k^2 + \|\tilde{\rho}\|_{k+1}^2\right) + \|\tilde{u}\|_{k+1}^2 \\ &\leq C_k \left(\|\tilde{w}\|_k^2 + \|\tilde{\rho}\|_{k+1}^2 + \|\tilde{f}_1\|_{k+1} + \|\tilde{f}\|_k^2\right), \end{aligned} \tag{4.18}$$

and

$$\|\tilde{w}\|^2_{k,[0,T]} \leq C_k(T) \left(\|\tilde{w}_0\|_k^2 + \|\tilde{\rho}_0\|_{k+1}^2 + \int_0^T \left(\|\tilde{f}(t)\|_k^2 + \|\tilde{f}_1(t)\|_{k+1}^2\right) dt\right). \tag{4.19}$$

Proof. Let ∇^k denote the derivative of order k with respect to x. Then, applying ∇^j to (4.6), we have

$$\tilde{L}(\tilde{w}) \nabla^j \tilde{w} = \nabla^j \tilde{f} - \left[\nabla^j, \tilde{L}\right] \tilde{w}. \tag{4.20}$$

Taking the inner product of (4.20) with $A_0 \nabla^j \tilde{w}$, we obtain

$$\langle \partial_t \nabla^j \tilde{w}, A_0 \nabla^j \tilde{w}\rangle + \langle (T_1 + T_2) \nabla^j \tilde{w}, A_0 \nabla^j \tilde{w}\rangle = -\langle A_1 \nabla^j \tilde{w}, A_0 \nabla^j \tilde{w}\rangle + \langle \nabla^j \tilde{f}, A_0 \nabla^j \tilde{w}\rangle$$
$$+ \langle [\nabla^j, \tilde{L}]\tilde{w}, A_0 \nabla^j \tilde{w}\rangle.$$

Now, we observe that

$$\langle A_1 \nabla^j \tilde{w}, A_0 \nabla^j \tilde{w}\rangle = \langle \left(u\partial_x \nabla^j \tilde{\rho} + \rho \partial_x \nabla^j \tilde{u}, p'\rho^{-1} \partial_x \nabla^j \tilde{\rho} + u \partial_x \nabla^j \tilde{u}\right), \left(\nabla^j \tilde{\rho}, \rho \nabla^j \tilde{u}\right)\rangle,$$

where we have

$$\begin{aligned} \langle \rho \partial_x \nabla^j \tilde{u}, \nabla^j \tilde{\rho}\rangle &= \int_{\mathbb{R}} \rho \left(\partial_x \nabla^j \tilde{u}\right) \nabla^j \tilde{\rho} dx \\ &= -\int_{\mathbb{R}} \left(\nabla^j \tilde{u}\right) \partial_x \left(\rho \nabla^j \tilde{\rho}\right) dx \\ &\leq \tilde{C} \|\tilde{\rho}\|_{j+1} \|\tilde{u}\|_j, \end{aligned}$$

and

$$\begin{aligned}\langle u\partial_x\nabla^j\tilde{u},\rho\nabla^j\tilde{u}\rangle &= \int_{\mathbb{R}} u\rho\left(\partial_x\nabla^j\tilde{u}\right)\nabla^j\tilde{u}dx\\ &= \frac{1}{2}\int_{\mathbb{R}} u\rho\partial_x\left(\nabla^j\tilde{u}\right)^2dx\\ &= -\frac{1}{2}\int_{\mathbb{R}}\left(\nabla^j\tilde{u}\right)^2\partial_x\left(u\rho\right)dx\\ &\leq C\|\tilde{u}\|_j^2.\end{aligned}$$

Then we can write

$$\begin{aligned}&-\langle A_1\nabla^j\tilde{w},A_0\nabla^j\tilde{w}\rangle+\langle\nabla^j\tilde{f},A_0\nabla^j\tilde{w}\rangle+\langle[\nabla^j,\tilde{L}]\tilde{w},A_0\nabla^j\tilde{w}\rangle\\ &\leq C_j\left(\|\tilde{w}\|_j^2+\|\tilde{\rho}\|_{j+1}^2+\|\tilde{f}\|_j^2+|\langle A_0\nabla^j\tilde{w},[\nabla^j,\tilde{L}]\tilde{w}\rangle|\right).\end{aligned}\tag{4.21}$$

And using a reasoning similar to the one used to obtain (4.13), we can write

$$\langle\partial_t\nabla^j\tilde{w},A_0\nabla^j\tilde{w}\rangle\geq\frac{1}{2}\partial_t\|\sqrt{A_0}\nabla^j\tilde{w}\|^2-C_j\|\tilde{w}\|_j^2.$$

Moreover,

$$\begin{aligned}\langle T_2\nabla^j\tilde{w},A_0\nabla^j\tilde{w}\rangle &= \langle(0,-\frac{4}{3}\mu\rho^{-1}\triangle\nabla^j\tilde{u}),(\nabla^j\tilde{\rho},\rho\nabla^j\tilde{u}\rangle\\ &= -\frac{4}{3}\mu\int_{\mathbb{R}}\left(\triangle\nabla^j\tilde{u}\right)\nabla^j\tilde{u}dx\\ &= \frac{4}{3}\mu\int_{\mathbb{R}}\left(\nabla^j\tilde{u}_x\right)^2dx\\ &= \frac{4}{3}\mu\|\nabla^{j+1}\tilde{u}\|^2.\end{aligned}$$

Then, using the above two expressions and (4.21) we obtain

$$\begin{aligned}&\partial_t\|\nabla^j\tilde{w}\|^2+\|\nabla^{j+1}\tilde{u}\|^2+\langle A_0\nabla^j\tilde{w},T_1\nabla\tilde{w}\rangle\\ &\leq C_j\left(\|\tilde{w}\|_j^2+\|\tilde{\rho}\|_{j+1}^2+\|\tilde{f}\|_j^2+|\langle A_0\nabla^j\tilde{w},[\nabla^j,\tilde{L}]\tilde{w}\rangle|\right).\end{aligned}\tag{4.22}$$

Before we proceed, we note that up to here in the proof we have used only the first term in left-hand side of (4.17). Now, let us consider the term

$$\begin{aligned}\langle A_0\nabla^j\tilde{w},T_1\nabla^j\tilde{w}\rangle &= \langle(\nabla^j\tilde{\rho},\rho\nabla^j\tilde{u}),(0,-\nu\triangle\nabla^{j+1}\tilde{\rho}\rangle\\ &= -\nu\int_{\mathbb{R}}\left(\triangle\nabla^{j+1}\tilde{\rho}\right)\rho\nabla^j\tilde{u}dx\\ &= \nu\int_{\mathbb{R}}\left(\triangle\nabla^j\tilde{\rho}\right)\left(\rho\nabla^j\tilde{u}_x+\rho_x\nabla^j\tilde{u}\right)dx\\ &= \nu\langle\triangle\nabla^j\tilde{\rho},\rho\nabla^j\tilde{u}_x\rangle+\nu\langle\triangle\nabla^j\tilde{\rho},\rho_x\nabla^j\tilde{u}\rangle.\end{aligned}$$

From the first equation in (4.6), we have

$$\rho\nabla^j\tilde{u}_x=-\left(\partial_t+u\partial_x\right)\nabla^j\tilde{\rho}-\nabla^j\tilde{f}_1-[\nabla^j,\tilde{L}_1(w)]\tilde{w},$$

and if we make the computation, we can see that

$$[\nabla^j,\tilde{L}_1(w)]\tilde{w}=\{\cdots\}_1+\{\cdots\}_2,$$

where $\{\cdots\}_1$ and $\{\cdots\}_2$ contain terms of the form

$$\alpha_j \left(\nabla^{j-i} u\right)\left(\nabla^{i+1} \tilde{\rho}\right),$$
$$\alpha_j \left(\nabla^{j-i} \rho\right)\left(\nabla^{i+1} \tilde{u}\right),$$

respectively, for $0 \leq i \leq j-1$ and some α_j's constants. Then we have

$$\begin{aligned}
\nu\langle \triangle \nabla^j \tilde{\rho}, \rho \nabla^j \tilde{u}_x\rangle &= \nu\langle \triangle \nabla^j \tilde{\rho}, -\left(\partial_t + u\partial_x\right)\nabla^j \tilde{\rho} - \nabla^j \tilde{f}_1 - \left(\{\cdots\}_1 + \{\cdots\}_2\right)\rangle \\
&\geq -\nu\langle \triangle \nabla^j \tilde{\rho}, \nabla^j \partial_t \tilde{\rho}\rangle - \frac{\nu}{2}\int_{\mathbb{R}} u\partial_x \left(\nabla^{j+1}\tilde{\rho}\right)^2 dx - C_j \|\nabla^{j+1}\tilde{\rho}\| \|\tilde{f}_1\|_{j+1} - C_j \|\nabla^{j+1}\tilde{\rho}\| \|\nabla^{j+1}\tilde{u}\| \\
&\geq -\nu\langle \triangle \nabla^j \tilde{\rho}, \nabla^j \partial_t \tilde{\rho}\rangle + \frac{\nu}{2}\int_{\mathbb{R}} \partial_x u \left(\nabla^{j+1}\tilde{\rho}\right)^2 dx - C_j \|\nabla^{j+1}\tilde{\rho}\| \|\tilde{f}_1\|_{j+1} - C_j \|\nabla^{j+1}\tilde{\rho}\| \|\nabla^{j+1}\tilde{u}\| \\
&\geq -\nu\langle \triangle \nabla^j \tilde{\rho}, \nabla^j \partial_t \tilde{\rho}\rangle - C_j \|\nabla^{j+1}\tilde{\rho}\|^2 - C_j \|\nabla^{j+1}\tilde{\rho}\| \|\tilde{f}_1\|_{j+1} - C_j \|\nabla^{j+1}\tilde{\rho}\| \|\nabla^{j+1}\tilde{u}\|.
\end{aligned}$$

In the above computations we used the second term in the left-hand side of (4.17). Also we have

$$\begin{aligned}
\nu\langle \triangle \nabla^j \tilde{\rho}, \rho_x \nabla^j \tilde{u}\rangle &= \nu \int_{\mathbb{R}} \left(\triangle \nabla^j \tilde{\rho}\right) \rho_x \nabla^j \tilde{u} dx \\
&= -\nu \int_{\mathbb{R}} \left(\nabla^{j+1}\tilde{\rho}\right) \partial_x \left(\rho_x \nabla^j \tilde{u}\right) dx \\
&\leq C_j \|\nabla^{j+1}\tilde{\rho}\| \|\nabla^{j+1}\tilde{u}\|.
\end{aligned}$$

Then we can write

$$\begin{aligned}
\langle A_0 \nabla^j \tilde{w}, T_1 \nabla^j \tilde{w}\rangle \geq \frac{\nu}{2}\partial_t \|\nabla^{j+1}\tilde{\rho}\|^2 - \epsilon \|\nabla^{j+1}\tilde{u}\|^2 \\
- C_j \left(\|\nabla^{j+1}\tilde{\rho}\|^2 + \|\tilde{f}_1\|_{j+1}^2\right),
\end{aligned} \tag{4.23}$$

where we have used that

$$\begin{aligned}
\nu\langle \triangle \nabla^j \tilde{\rho}, \nabla \tilde{\rho}_t\rangle = \nu \int_{\mathbb{R}} \left(\triangle \nabla^j \tilde{\rho}\right)\left(\nabla^j \tilde{\rho}_t\right) dx &= -\nu \int_{\mathbb{R}} \left(\nabla^{j+1}\tilde{\rho}\right)\left(\nabla^{j+1}\tilde{\rho}_t\right) dx \\
&= -\frac{\nu}{2}\partial_t \int_{\mathbb{R}} \left(\nabla^{j+1}\tilde{\rho}\right)^2 dx \\
&= -\frac{\nu}{2}\partial_t \|\nabla^{j+1}\tilde{\rho}\|^2.
\end{aligned}$$

Now, we discuss the terms involving the commutator $[\nabla^j, \tilde{L}]$. By the form of $[\nabla^j, \tilde{L}_1(w)]\tilde{w}$ we have

$$\langle \nabla^j \tilde{\rho}, [\nabla^j, \tilde{L}_1(w)]\tilde{w}\rangle \leq C_j \|\tilde{w}\|_j^2,$$

and for the second coordinate of the commutator we obtain

$$[\nabla^j, \tilde{L}_2(\tilde{w})] = \{\cdots\}_a + \{\cdots\}_b + \{\cdots\}_c,$$

where $\{\cdots\}_a$, $\{\cdots\}_b$, and $\{\cdots\}_c$ contain terms of the form

$$\beta_j \left(\nabla^{j-i} u\right)\left(\nabla^{i+1}\tilde{u}\right),$$
$$\beta_j \left(\nabla^{j-i}\left(p'(\rho)\rho^{-1}\right)\right)\left(\nabla^{i+1}\tilde{\rho}\right), \text{ and}$$
$$\beta_j \left(\nabla^{j-i}\rho^{-1}\right)\left(\nabla^{i+2}\tilde{u}\right),$$

respectively, for $0 \leq i \leq j-1$, and some β_j's constants. Then we can write

$$\langle \rho \nabla^j \tilde{u}, [\nabla^j, \tilde{L}_2(w)]\tilde{w}\rangle \leq C_j \left(\|\tilde{w}\|_j^2 + \|\nabla^{j+1}\tilde{\rho}\|^2\right) + \epsilon \|\nabla^{j+1}\tilde{u}\|^2.$$

Thus, with the last two estimates, we obtain

$$|\langle A_0\nabla^j\tilde{w},[\nabla^j,\tilde{L}(w)]\tilde{w}\rangle|\le C_j\left(\|\tilde{w}\|_j^2+\|\nabla^{j+1}\tilde{\rho}\|^2\right)+\epsilon\|\nabla^{j+1}\tilde{u}\|^2. \tag{4.24}$$

Combining (4.22)-(4.24) and taking $j=k$, we obtain

$$\begin{aligned}&\partial_t\left(\|\tilde{w}\|_k^2+\|\tilde{\rho}\|_{k+1}^2\right)+\|\tilde{u}\|_{k+1}^2\\ &\le C_k\left(\|\tilde{w}\|_k^2+\|\tilde{\rho}\|_{k+1}^2+\|\tilde{f}_1\|_{k+1}+\|\tilde{f}\|_k^2\right),\end{aligned}$$

which is (4.18). Applying Gronwall's lemma to last expression, as we did to obtain (4.10), we obtain (4.19). ■

4.2 Existence of solutions for the linearized problem

By the energy estimate (4.19) we need only to prove the existence of the solution for the problem (4.4)-(4.5) for $\tilde{f}\in C_0^\infty([0,T]\times\mathbb{R})$ and $\tilde{w}=0$. Let the sequence of smooth functions $\{(\tilde{f})_i\}$ converge to $\tilde{f}$. Then, by (4.19), the solution sequence $\{\tilde{w}_i\}$ converges to the desired function. The homogeneous initial condition is due to the linearity of the operator $\tilde{L}$.

Remark: It is well known that $C_0^\infty\left([0,T];H^k(\mathbb{R})\right)$ is dense in $L^2\left([0,T];H^k(\mathbb{R})\right)$ (see Theorem 2.1 in [11], Chapter 1)). Also, since the space $C_0^\infty(\mathbb{R})$ is dense in $H^k(\mathbb{R})$, and we can think the elements of $C_0^\infty([0,T]\times\mathbb{R})$ as elements of $C_0^\infty\left([0,T];H^k(\mathbb{R})\right)$, then we can approximate functions $g\in C_0^\infty\left([0,T];H^k(\mathbb{R})\right)$ by functions in $C_0^\infty([0,T]\times\mathbb{R})$. Thus, we conclude that any function $f\in L^2\left([0,T];H^k(\mathbb{R})\right)$, can be approximated by functions $h\in C_0^\infty([0,T]\times\mathbb{R})$.

In the sequel, we use the dual method to prove the existence of the following problem in $[0,T]$:

$$L\tilde{w}\equiv\partial_t\tilde{w}+A_1\partial_x\tilde{w}+(T_1+T_2)\,\tilde{w}=\tilde{f}, \tag{4.25}$$

$$\tilde{w}(x,0)=0, \tag{4.26}$$

where A_1 and the operator T_1, T_2 are defined in (4.7) and (4.8), respectively. The adjoint operator L^* for (4.25) is defined by

$$\langle L\tilde{w},\tilde{\phi}\rangle=\langle\tilde{w},L^*\tilde{\phi}\rangle,$$

where the above equality has to be understood in the sense of the pairing of $L^2\left([0,T];H^{-k}(\mathbb{R})\right)$ and $L^2\left(0,T;H^k(\mathbb{R})\right)$. Then to prove the existence of weak solutions $\tilde{w}\in L^2\left([0,T];H^k(\mathbb{R})\right)$ for (4.25) and (4.26), we need to establish the energy estimates of negative order for the operator L^*. We first derive the classical energy estimates for

$$L^*\tilde{\phi}=-\partial_t\tilde{\phi}-A_1^*\partial_x\tilde{\phi}+(T_1^*+T_2^*)\,\tilde{\phi}=\tilde{g}, \tag{4.27}$$

$$\tilde{\phi}(x,T)=0. \tag{4.28}$$

Theorem 4.3 *The solutions of (4.27) and (4.28) satisfy the following estimate:*

$$\|\tilde{\phi}(t)\|^2+\|\partial_x\tilde{\phi}_2(t)\|^2\le C\int_0^T\|\tilde{g}(\tau)\|^2d\tau. \tag{4.29}$$

Proof. The operator L^* can be explicitly written as follows:

$$\begin{cases}-(\partial_t+\partial_x u)\,\tilde{\phi}_1-\partial_x\left(p'(\rho)\rho^{-1}\tilde{\phi}_2\right)+\nu\triangle\partial_x\tilde{\phi}_2=\tilde{g}_1,\\ -(\partial_t+\partial_x u)\,\tilde{\phi}_2-\partial_x\left(\rho\tilde{\phi}_1\right)-\frac{4}{3}\partial_{xx}\left(\rho^{-1}\tilde{\phi}_2\right)=\tilde{g}_2.\end{cases} \tag{4.30}$$

Taking the inner product of the second equation in (4.30) with $\tilde{\phi}_2$, we have

$$\left\langle -\partial_t \tilde{\phi}_2, \tilde{\phi}_2 \right\rangle - \frac{4}{3}\mu \left\langle \partial_{xx}\left(\rho^{-1}\tilde{\phi}_2\right), \tilde{\phi}_2 \right\rangle = \left\langle \partial_x \left(\rho\tilde{\phi}_1 + u\tilde{\phi}_2\right) + \tilde{g}_2, \tilde{\phi}_2 \right\rangle.$$

We integrate by parts the second therm in the left-hand side of the last expression to obtain

$$\begin{aligned}
-\left\langle \partial_{xx}\left(\rho^{-1}\tilde{\phi}_2\right), \tilde{\phi}_2 \right\rangle &= -\int_{\mathbb{R}} \partial_{xx}\left(\rho^{-1}\tilde{\phi}_2\right)\tilde{\phi}_2 dx \\
&= \int_{\mathbb{R}} \partial_x \left(\rho^{-1}\tilde{\phi}_2\right)\partial_x\tilde{\phi}_2 dx \\
&= \int_{\mathbb{R}} \partial_x \left(\rho^{-1}\right)\tilde{\phi}_2\partial_x\tilde{\phi}_2 dx + \int_{\mathbb{R}} \rho^{-1}\partial_x\tilde{\phi}_2\partial_x\tilde{\phi}_2 dx \\
&= \|\rho^{-1/2}\tilde{\phi}_x\|^2 + \frac{1}{2}\int_{\mathbb{R}} \partial_x\left(-\rho^{-1}\right)\partial_x\left(\tilde{\phi}_2\right)^2 dx \\
&= \|\rho^{-1/2}\tilde{\phi}_x\|^2 + \frac{1}{2}\int_{\mathbb{R}} \left(\partial_{xx}\rho^{-1}\right)\tilde{\phi}_2^2 dx \\
&\geq \|\rho^{-1/2}\tilde{\phi}_x\|^2 - C\|\tilde{\phi}_2\|^2.
\end{aligned}$$

Also

$$\left\langle -\partial_t\tilde{\phi}_2, \tilde{\phi}_2 \right\rangle = -\int_{\mathbb{R}} \left(\partial_t\tilde{\phi}_2\right)\tilde{\phi}_2 dx = -\frac{1}{2}\partial_t \int_{\mathbb{R}} \tilde{\phi}_2^2 dx = -\frac{1}{2}\partial_t\|\tilde{\phi}_2\|^2.$$

We can estimate the terms in the right-hand side as

$$\begin{aligned}
\langle \partial_x(u\tilde{\phi}_2), \tilde{\phi}_2 \rangle &= \int_{\mathbb{R}} \partial_x(u\tilde{\phi}_2)\tilde{\phi}_2 dx \\
&= -\int_{\mathbb{R}} u\tilde{\phi}_2\partial_x\tilde{\phi}_2 dx \\
&= -\frac{1}{2}\int_{\mathbb{R}} u\partial_x\left(\tilde{\phi}_2^2\right) dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \left(\partial_x u\right)\tilde{\phi}_2^2 dx \\
&\leq C\|\tilde{\phi}_2\|^2,
\end{aligned}$$

and

$$\begin{aligned}
\langle \partial_x(\rho\tilde{\phi}_1), \tilde{\phi}_2 \rangle &= \int_{\mathbb{R}} \tilde{\phi}_2\partial_x\left(\rho\tilde{\phi}_1\right) dx \\
&= -\int_{\mathbb{R}} \rho^{3/2}\tilde{\phi}_1\rho^{-1/2}\partial_x\tilde{\phi}_2 dx \\
&\leq C\|\tilde{\phi}_1\|^2 + \frac{1}{3}\mu\|\rho^{-1/2}\partial_x\tilde{\phi}_2\|^2.
\end{aligned}$$

Here we have used the inequality

$$\|a\|\|b\| \leq \epsilon\|a\|^2 + \frac{1}{\epsilon}\|b\|^2, \tag{4.31}$$

for $\epsilon > 0$. Thus we get the following estimate

$$-\frac{1}{2}\partial_t\|\tilde{\phi}_2\|^2 + \mu\|\rho^{-1}\partial_x\tilde{\phi}_2\|^2 \leq C\left(\|\tilde{\phi}\|^2 + \|\tilde{g}_2\|^2\right). \tag{4.32}$$

Now, we take the inner product of the second equation in (4.30) with $-\partial_{xx}\tilde{\phi}_2$ to obtain

$$\left\langle \partial_t\tilde{\phi}_2, \partial_{xx}\tilde{\phi}_2 \right\rangle + \frac{4}{3}\mu\left\langle \partial_{xx}\left(\rho^{-1}\tilde{\phi}_2\right), \partial_{xx}\tilde{\phi}_2 \right\rangle + \langle \partial_x\left(\rho\tilde{\phi}_1\right), \partial_{xx}\tilde{\phi}_2 \rangle = \left\langle \partial_x(u\tilde{\phi}_2) + \tilde{g}_2, -\partial_{xx}\tilde{\phi}_2 \right\rangle.$$

And for the second term in the left-hand side, we have the estimate

$$\begin{aligned}\langle \partial_{xx}\left(\rho^{-1}\tilde{\phi}_2\right), \partial_{xx}\tilde{\phi}_2\rangle &= \int_{\mathbb{R}} \partial_{xx}\left(\rho^{-1}\tilde{\phi}_2\right)\partial_{xx}\tilde{\phi}_2 dx \\ &= \int_{\mathbb{R}} \left\{\tilde{\phi}_2\partial_{xx}\rho^{-1} + 2\left(\partial_x\rho^{-1}\right)\left(\partial_x\tilde{\phi}_2\right) + \rho^{-1}\partial_{xx}\tilde{\phi}_2\right\}\partial_{xx}\tilde{\phi}_2 dx \\ &= \|\rho^{-1/2}\partial_{xx}\tilde{\phi}_2\|^2 + \int_{\mathbb{R}} \partial_x\left(\rho^{-1}\right)\partial_x\left(\partial_x\tilde{\phi}_2\right)^2 dx + \int_{\mathbb{R}} \rho^{1/2}\tilde{\phi}_2\rho^{-1/2}\left(\partial_{xx}\tilde{\phi}_2\right)\partial_{xx}\left(\rho^{-1}\right)dx \\ &\geq \|\rho^{-1/2}\triangle\tilde{\phi}_2\|^2 - \int_{\mathbb{R}}\left(\partial_{xx}\rho^{-1}\right)\left(\partial_x\tilde{\phi}_2\right)^2 dx - C\|\tilde{\phi}_2\|^2 - \epsilon\|\rho^{-1/2}\partial_{xx}\tilde{\phi}_2\| \\ &\geq (1-\epsilon)\|\rho^{-1/2}\triangle\tilde{\phi}_2\|^2 - C\left(\|\tilde{\phi}_2\|^2 + \|\partial_x\tilde{\phi}_2\|\right),\end{aligned}$$

where, again, we have used (4.31). Also, we have

$$\begin{aligned}\langle \partial_t\tilde{\phi}_2, \partial_{xx}\tilde{\phi}_2\rangle &= \int_{\mathbb{R}}\left(\partial_t\tilde{\phi}_2\right)\partial_{xx}\tilde{\phi}_2 dx \\ &= -\int_{\mathbb{R}}\left(\partial_t\partial_x\tilde{\phi}_2\right)\partial_x\tilde{\phi}_2 dx \\ &= -\frac{1}{2}\partial_t\int_{\mathbb{R}}\left(\partial_x\tilde{\phi}_2\right)^2 dx \\ &= -\frac{1}{2}\partial_t\|\partial_x\tilde{\phi}_2\|^2,\end{aligned}$$

and

$$\langle \partial_x(\rho\tilde{\phi}_1), \triangle\tilde{\phi}_2\rangle = -\langle \rho\tilde{\phi}_1, \triangle(\partial_x\tilde{\phi}_2)\rangle.$$

Using an argument similar to the one we used in obtaining (4.32), we get

$$\left\langle \tilde{g}_2, -\partial_{xx}\tilde{\phi}_2\right\rangle \leq C\left(\|\tilde{\phi}\|^2 + \|\partial_x\tilde{\phi}_2\|^2 + \|\tilde{g}_2\|^2\right) + \epsilon\|\rho^{-1/2}\partial_{xx}\tilde{\phi}_2\|^2.$$

Moreover,

$$\begin{aligned}-\langle \partial_x(u\tilde{\phi}_2), \partial_{xx}\tilde{\phi}_2\rangle &= -\langle (\partial_x u)\tilde{\phi}_2, \partial_{xx}\tilde{\phi}_2\rangle - \langle u\partial_x\tilde{\phi}_2, \partial_{xx}\tilde{\phi}_2\rangle \\ &= -\int_{\mathbb{R}}(\partial_x u)\tilde{\phi}_2\partial_{xx}\tilde{\phi}_2 dx - \frac{1}{2}\int_{\mathbb{R}} u\partial_x\left(\partial_x\tilde{\phi}_2\right)^2 dx \\ &= \int_{\mathbb{R}}\partial_x((\partial_x u)\tilde{\phi}_2)\partial_x\tilde{\phi}_2 dx + \frac{1}{2}\int_{\mathbb{R}}(\partial_x u)\left(\partial_x\tilde{\phi}_2\right)^2 dx \\ &\leq C\left(\|\tilde{\phi}_2\|^2 + \|\partial_x\tilde{\phi}_2\|^2\right).\end{aligned}$$

Thus, we obtain the estimate

$$\begin{aligned}-\frac{1}{2}\partial_t\|\partial_x\tilde{\phi}_2\|^2 + \mu\|\rho^{-1/2}\triangle\tilde{\phi}_2\|^2 - \langle \rho\tilde{\phi}_1, \triangle(\partial_x\tilde{\phi}_2)\rangle \\ \leq C\left(\|\tilde{\phi}_2\|^2 + \|\partial_x\tilde{\phi}_2\|^2 + \|\tilde{g}_2\|^2\right).\end{aligned} \tag{4.33}$$

From the first equation in (4.30) one finds that

$$\triangle\partial_x\tilde{\phi}_2 = \nu^{-1}\left[\left(\partial_t + \partial_x u\right)\tilde{\phi}_1 + \partial_x(p'(\rho)\rho^{-1}\tilde{\phi}_2) + \tilde{g}_1\right],$$

and

$$\begin{aligned}\langle \rho\tilde{\phi}_1, \partial_t\tilde{\phi}_1\rangle = \int_{\mathbb{R}}\rho\tilde{\phi}_1\partial_t\tilde{\phi}_1 dx &= \frac{1}{2}\partial_t\|\rho^{1/2}\tilde{\phi}_1\|^2 - \frac{1}{2}\int_{\mathbb{R}}\tilde{\phi}_1^2\partial_t\rho dx \\ &\leq \frac{1}{2}\partial_t\int_{\mathbb{R}}\rho\tilde{\phi}_1^2 dx + C\|\tilde{\phi}_1\|^2,\end{aligned}$$

$$\begin{aligned}\langle \rho\tilde{\phi}_1, \partial_x(u\tilde{\phi}_1)\rangle &= \langle \rho\tilde{\phi}_1, (\partial_x u)\tilde{\phi}_1 + u\partial_x\tilde{\phi}_1\rangle \\ &= \int_{\mathbb{R}} \rho(\partial_x u)\tilde{\phi}_1^2 dx + \frac{1}{2}\int_{\mathbb{R}} \rho u \partial_x \left(\tilde{\phi}_1\right)^2 dx \\ &= \int_{\mathbb{R}} \rho(\partial_x u)\tilde{\phi}_1^2 - \frac{1}{2}\int_{\mathbb{R}} (\partial_x(\rho u))\tilde{\phi}_1^2 dx \\ &\le C\|\tilde{\phi}_1\|^2,\end{aligned}$$

$$\begin{aligned}\langle \rho\tilde{\phi}_1, \partial_x(p'(\rho)\rho^{-1}\tilde{\phi}_2)\rangle &= \int_{\mathbb{R}} \rho\tilde{\phi}_1\partial_x(p'(\rho)\rho^{-1}\tilde{\phi}_2)dx \\ &\le C\left(\|\tilde{\phi}_1\|^2 + \|\tilde{\phi}_2\|^2 + \|\partial_x\tilde{\phi}_2\|^2\right) \\ &= C\left(\|\tilde{\phi}\|^2 + \|\partial_x\tilde{\phi}_2\|^2\right).\end{aligned}$$

Therefore, from (4.33) we obtain the following estimate:

$$-\partial_t\left(\|\tilde{\phi}_1\|^2 + \|\partial_x\tilde{\phi}_2\|^2\right) + \|\rho^{-1/2}\triangle\tilde{\phi}_2\|^2 \le C\left(\|\tilde{\phi}\|^2 + \|\partial_x\tilde{\phi}_2\|^2 + \|\tilde{g}\|^2\right). \tag{4.34}$$

Combining (4.32) and (4.34) yields

$$-\partial_t\left(\|\tilde{\phi}\|^2 + \|\partial_x\tilde{\phi}_2\|^2\right) + \|\rho^{-1/2}\triangle\tilde{\phi}_2\|^2 \le C\left(\|\tilde{\phi}\|^2 + \|\partial_x\tilde{\phi}_2\|^2 + \|\tilde{g}\|^2\right),$$

from which it follows that

$$-\partial_t\left(\|\tilde{\phi}\|^2 + \|\partial_x\tilde{\phi}_2\|^2\right) \le C\left(\|\tilde{\phi}\|^2 + \|\partial_x\tilde{\phi}_2\|^2 + \|\tilde{g}\|^2\right). \tag{4.35}$$

Applying the Gronwall inequality and noticing that $\tilde{\phi} = 0$ in $t = T$, we get (4.29). ■

Next, we derive the negative norm estimates for the solutions of (4.27) and (4.28). Let Λ denote the operator with symbol

$$\lambda(\xi) = \sqrt{1+\xi^2},$$

where ξ is the dual variable of x. Now, we state the following theorem

Theorem 4.4 *For any $s \in \mathbb{R}$, the solution of (4.27) and (4.28) satisfies the following estimate:*

$$\|\Lambda^s\tilde{\phi}(t)\|^2 + \|\Lambda^{s+1}\tilde{\phi}_2\|^2 \le C\int_0^T \|\Lambda^s\tilde{g}(\tau)\|^2 d\tau. \tag{4.36}$$

We omit the proof, since its proof is very similar to the one of the previous theorem, except that we need to take a little care with the pseudo-differential operator Λ. Using theorem 4.4 we can derive the existence of a differentiable weak solution for (4.25) and (4.26). Since, for any large integer k, we have

$$\left|\int_0^T \langle\tilde{f}, \tilde{\phi}\rangle dt\right| = \left|\int_0^T \langle\Lambda^k\tilde{f}, \Lambda^{-k}\tilde{\phi}\rangle dt\right| \le \int_0^T \|\Lambda^k\tilde{f}\|\|\Lambda^{-k}\tilde{\phi}\|dt,$$

and applying (4.36) with $s = -k$, we obtain

$$\left|\int_0^T \langle\tilde{f}, \tilde{\phi}\rangle dt\right| \le C\left(\int_0^T \|\Lambda^k\tilde{f}\|dt\right)\left(\int_0^T \|\Lambda^{-k}L^*\tilde{\phi}\|^2 dt\right)^{1/2}.$$

Therefore,

$$\int_0^T \langle\tilde{f}, \tilde{\phi}\rangle dt,$$

defines a bounded linear functional of $L^*\tilde{\phi}$ in the space $L^2\left([0,T];H^{-k}(\mathbb{R})\right)$. Then, using the Hanh-Banach extension theorem and the Riesz representation theorem, we conclude that there exist a unique $\tilde{w} \in L^2\left([0,T];H^k(\mathbb{R})\right)$ such that

$$\int_0^T \langle \tilde{f}, \tilde{\phi}\rangle dt = \int_0^T \langle \tilde{w}, L^*\tilde{\phi}\rangle dt = \int_0^T \langle L\tilde{w}, \tilde{\phi}\rangle dt \quad \forall \tilde{\phi} \in C_0^\infty\left([0,T]\times\mathbb{R}\right), \tag{4.37}$$

since it is well known that the dual space of $L^p\left([0,T];X\right)$ is $L^q\left([0,T];X^*\right)$, with (p,q) conjugate indices. We note that the last equality is, precisely, the definition of L^*. Now, since equality (4.37) is taking place in $L^2\left([0,T];H^{-k}(\mathbb{R})\right)$, we can conclude that

$$\langle \tilde{f}, \phi\rangle_{L^2(\mathbb{R})} = \langle L\tilde{w}, \phi\rangle_{L^2(\Omega)}, \quad \forall \phi \in C_0^\infty\left(\mathbb{R}\right),$$

for almost every $t \in [0,T]$ (see Lemma 7.4 in [12]). Also, we are assuming $k \geq 3$, so that by the embedding Sobolev theorem, the space derivatives in the above expression are derivatives in the classical sense. Then, with all the previous discussion and some remarks that are going to be make in the next subsection, we have proved the following:

Theorem 4.5 *For all $\tilde{f} \in L^2\left([0,T];H^k\left(\mathbb{R}\right)\right)$, $\tilde{f}_1 \in L^2\left([0,T];H^{k+1}\left(\mathbb{R}\right)\right)$, $\tilde{w}_0 \in H^k\left(\mathbb{R}\right)$, $\tilde{\rho}_0 \in H^{k+1}\left(\mathbb{R}\right)$, the Cauchy problem (4.4)-(4.5) has a unique solution $\tilde{w}$ such that its norm $\|\tilde{w}\|_{k,[0,T]}$ is bounded and satisfies the estimate (4.19).*

4.3 Local existence of solution for the nonlinear problem

Once that we have established the existence of the linearized problem, we are in a position to prove the local existence of solutions for the Cauchy problem of (4.1) and (4.3). Let us consider the initial value problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2\right)_x = \left\{-p - \frac{\nu}{2}\rho_x^2 + \nu\rho\rho_{xx} + \frac{4}{3}\mu u_x\right\}_x, \end{cases} \tag{4.38}$$

$$\rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x). \tag{4.39}$$

We have the following

Theorem 4.6 *For any initial condition (ρ_0, u_0) such that $\rho_0 - \bar{\rho}_0 \geq \delta > 0$, $\rho_0 - \bar{\rho}_0 \in H^{k+1}\left(\mathbb{R}\right)$, and $(\rho_0 - \bar{\rho}_0, u_0) \in \left(H^k\left(\mathbb{R}\right)\right)^2$, where $\bar{\rho}_0 > 0$ is a constant, there exists a $T > 0$ such that for $t \in [0,T]$, the Cauchy problem (4.38) and (4.39) has a unique solution (ρ, u) such that $\rho - \bar{\rho}_0 \in L^\infty\left([0,T];H^{k+1}\left(\mathbb{R}\right)\right)$, $u \in L^\infty\left([0,T];H^k\left(\mathbb{R}\right)\right)$, and*

$$\|w - (\bar{\rho}_0, 0)\|_{k,[0,T]}^2 \leq C_k\left(\|(\rho_0 - \bar{\rho}_0, u_0)\|_k^2 + \|\rho_0 - \bar{\rho}_0\|_{k+1}^2\right).$$

Remark. The solution of the above theorem are classical solutions, i.e., all the derivatives in (4.38) exist and are continuous, and (4.38) and (4.39) are satisfied in the classical sense.

The remark is shown as follows. First, from

$$\rho \in L^\infty\left([0,T];H^5\left(\mathbb{R}\right)\right), \quad u \in L^\infty\left([0,T];H^4\left(\mathbb{R}\right)\right), \tag{4.40}$$

we have

$$\partial_x^3\rho \in L^\infty\left([0,T];H^2\left(\mathbb{R}\right)\right), \quad \partial_x^2 u \in L^\infty\left([0,T];H^2\left(\mathbb{R}\right)\right). \tag{4.41}$$

Since in the first equation in (4.38) we have only fist order spatial derivatives, and in the second we have up to third order, we obtain

$$\partial_t\rho \in L^\infty\left([0,T];H^3\left(\mathbb{R}\right)\right), \quad \partial_t u \in L^\infty\left([0,T];H^2\left(\mathbb{R}\right)\right). \tag{4.42}$$

Therefore,

$$\partial_t \partial_x^3 \rho \in L^\infty\left([0,T]; H^0(\mathbb{R})\right), \quad \partial_t \partial_x^2 u \in L^\infty\left([0,T]; H^0(\mathbb{R})\right). \tag{4.43}$$

From (4.41) and (4.43), and the fact that $L^\infty([0,T];X) \subset L^2([0,T];X)$ for $T>0$ finite (and Theorem 4 in [6], p. 304), we have

$$\partial_x^3 \rho \in C\left([0,T]; H^1(\mathbb{R})\right), \quad \partial_x^2 u \in C\left([0,T]; H^1(\mathbb{R})\right).$$

Thus, by Sobolev's imbedding theorem,

$$\partial_x^3 \rho \in C([0,T]; C_b(\mathbb{R})), \quad \partial_x^2 u \in C([0,T]; C_b(\mathbb{R})), \tag{4.44}$$

where $C_b(\mathbb{R})$ denotes the bounded and continuous real functions defined on $\mathbb{R}$. Again from (4.38), we obtain

$$\partial_t^2 \rho \in L^\infty\left([0,T]; H^1(\mathbb{R})\right), \quad \partial_t^2 u \in L^\infty\left([0,T]; H^0(\mathbb{R})\right). \tag{4.45}$$

Them combining (4.42) and (4.45), by the same reasoning used to get (4.45), we have

$$\partial_t \rho \in C([0,T]; C_b(\mathbb{R})), \quad \partial_t u \in C([0,T]; C_b(\mathbb{R})). \tag{4.46}$$

This concludes the proof of the remark.

We write (4.38) in quasilinear form, that is

$$\mathcal{L}(w)\,w \equiv \begin{cases} (\partial_t + u\partial_x)\,\rho + \rho u_x = 0, \\ (\partial_t + u\partial_x)\,u + p'(\rho)\,\rho^{-1}\partial_x \rho - \nu \triangle \partial_x \rho - \frac{4}{3}\mu \rho^{-1} \triangle u = 0. \end{cases}$$

Now, let us consider the following problem

$$\begin{cases} \mathcal{L}(w)\,w = f, \\ w(x,0) = (\rho_0 - \bar{\rho}_0, u_0), \end{cases} \tag{4.47}$$

where f is such that $f_1 \in L^2\left([0,T]; H^{k+1}(\mathbb{R})\right)$ and $f_2 \in L^2\left([0,T]; H^k(\mathbb{R})\right)$. Then, Theorem 4.6 is equivalent to the following

Theorem 4.7 *Under the conditions of Theorem 4.6, for any $f_1 \in L^2\left([0,T]; H^{k+1}(\mathbb{R})\right)$ and $f_2 \in L^2\left([0,T]; H^k(\mathbb{R})\right)$, there exists a $T>0$ such that in $t \in [0,T]$ the Cauchy problem (4.47) has a unique solution w such that $\rho - \bar{\rho}_0 \in L^\infty\left([0,T]; H^{k+1}(\mathbb{R})\right)$, $u \in L^\infty\left([0,T]; H^k(\mathbb{R})\right)$, satisfying*

$$\|w - (\bar{\rho}_0, 0)\|_{k,[0,T]}^2 \le C_k \left(\|(\rho_0 - \bar{\rho}_0, u_0)\|_k^2 + \|\rho_0 - \bar{\rho}_0\|_{k+1}^2 + \int_0^T \left(\|f\|_k^2 + \|f_1\|_{k+1}^2 \right) dt \right). \tag{4.48}$$

Before we start the proof of the theorem, let us just mention that the estimate (4.48) is the same as in Theorem 4.6, just by taking $f_1 = f_2 = 0$.

Proof. We are going to use a fixed point argument. Let $w_0(x,t) = 0$ and $w_j(x,t)$ $(j = 1,2,\ldots)$ be defined as the unique solution (which exists because of Theorem 4.5) of the following linear Cauchy problem:

$$\mathcal{L}(w_{j-1})\,(w_j - (\bar{\rho}_0, 0)) = f, \quad w_j(x,0) = (\rho_0 - \bar{\rho}_0, u_0). \tag{4.49}$$

The proof finishes if we can prove that the above problem has a fixed point. First, applying (4.19), we obtain for $v_1 = w_1 - (\bar{\rho}_0, 0)$

$$\|v_1\|_{k,[0,T]}^2 \le C_k(T) \left(\|(\rho_0 - \bar{\rho}_0, u_0)\|_k^2 + \|\rho_0 - \bar{\rho}_0\|_{k+1}^2 + \int_0^T \left(\|f\|_k^2 + \|f_1\|_{k+1}^2 \right) dt \right), \tag{4.50}$$

and by Sobolev imbedding, the norm $\| \cdot \|_{k,[0,T]}$ is an upper bound of the supremum norm in (x,t). Then, with the last estimate, we choose $T \ll 1$ such that $\tilde{\rho}_1 > 0$ and bounded from below, so that

$$\sup_{x,t} \left(\tilde{\rho}_1\right)^{-1},$$

is well defined. Also, by (4.46) applied to v_1, and again by the Sobolev imbedding theorem, to obtain (4.17) for w_1 it is enough to have

$$\|v_1\|^2_{k,[0,T]} \leq \beta_k,$$

which will be satisfied uniformly for all $j = 1, 2, \ldots$, and some constant $\delta > 0$

$$\|v_j\|^2_{k,[0,T]} \leq \delta \leq \beta_k, \tag{4.51}$$

because of the estimate (4.50). Notice that the constant C_k is the same for all iterations (since it only depends on β_k). The constant $\delta > 0$ depends on $T > 0$.

If we can show that the successive solutions v_j satisfy

$$\|v_j - v_{j-1}\|_{k-2,[0,T]} \leq \frac{1}{2}\|v_{j-1} - v_{j-2}\|_{k-2,[0,T]}, \tag{4.52}$$

by the fixed point theorem, applied to the space whose elements satisfy $\| \cdot \|_{k-2,[0,T]} < \infty$, we can conclude the proof. The above mentioned space is a Banach space because is the intersection of spaces of the form $L^\infty([0,T]; X)$ and $L^2([0,T]; Y)$, with X and Y being Banach spaces.

Let us prove (4.52) by induction. Assume that (4.51) and (4.52) are true for some j and all smaller indices. From the energy estimate (4.19) for the linearized problem, we obtain

$$\|v_{j+1}\|_{k,[0,T]} \leq C_k \left(\|(\rho_0 - \bar{\rho}_0, u_0)\|^2_k + \|\rho_0 - \bar{\rho}_0\|^2_{k+1} + \int_0^T \left(\|f\|^2_k + \|f_1\|^2_{k+1}\right) dt \right).$$

On the other hand, $w_{j+1} - w_j$ satisfies the homogeneous initial data and the equation:

$$\mathcal{L}(w_j)\left(w_{j+1} - w_j\right) = \left(\mathcal{L}(w_{j-1}) - \mathcal{L}(w_j)\right) w_j, \tag{4.53}$$

and if we make the computations, we obtain

$$\left(\mathcal{L}(w_{j-1}) - \mathcal{L}(w_j)\right) w_j = \begin{cases} \left(u_{j-1} - u_j\right) \partial_x \rho_j + \left(\rho_{j-1} - \rho_j\right) \partial_x u_j, \\ \left(u_{j-1} - u_j\right) \partial_x u_j + \left(p'(\rho_{j-1})\rho_{j-1}^{-1} - p'(\rho_j)\rho_j^{-1}\right) \partial_x \rho_j - \frac{4}{3}\mu \left(\rho_{j-1}^{-1} - \rho_j^{-1}\right) \partial_{xx} u_j. \end{cases} .$$

Since we have derivatives up to one and second order in the first and second term, respectively, then we can apply Theorem 4.2 for $k-2$, to obtain

$$\|w_{j+1} - w_j\|^2_{k-2,[0,T]} \leq C_{k-2}\delta \int_0^T \|w_j - w_{j-1}\|^2_{k-2} dt \leq \tilde{C}_{k-2}\delta \|w_j - w_{j-1}\|^2_{k-2,[0,T]}, \tag{4.54}$$

where we have used the Cauchy-Schwartz inequality in the left hand side of the estimate (4.19). Now, since

$$\|v_j - v_{j-1}\|_{k-2,[0,T]} = \|w_j - w_{j-1}\|_{k-2,[0,T]},$$

choosing δ such that $\tilde{C}_{k-2}\delta < \frac{1}{4}$, we get (4.52). This concludes the proof of Theorem 4.7. ■

5 Linear decay rates of solutions

Once we have seen that our model of interest is well-posed and we can find a compensation matrix, we are in position of obtaining decay rates of the solutions to the linearized system around a constant state (Maxwellian). This is the first step to give a nonlinear result.
First, we observe that the model in (3.10), linearized around a constant equilibrium state $(\bar{\tau}, \bar{u})$ for the specific volume and velocity field, respectively, and only taking the linear part, has the following structure in the frequency space:

$$\partial_t \hat{U} + i\xi A(\xi)\hat{U} + \xi^2 B\hat{U} = 0, \tag{5.1}$$

for $\xi \in \mathbb{R}$, $t > 0$ and where

$$A(\xi) = \begin{pmatrix} 0 & -1 \\ -\beta(\xi) & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & \bar{\mu}/\bar{\tau} \end{pmatrix}.$$

Here $\bar{\tau} > c_0 > 0$ (no vacuum), $\bar{\mu} = \mu(\bar{u})$ (kinematic viscosity), and

$$\beta(\xi) = \bar{q} + \xi^2 \bar{k} > 0,$$

with $\bar{k} = k(\bar{\tau}) > 0$ (capillarity coefficient), and $\bar{q} = -p'(\bar{\tau}) > 0$ (a positive constant, since we are assuming the pressure function satisfies $p'(\tau) < 0$ for all τ). $\hat{U}$ is the Fourier transform of $(\tau, u)^T$, perturbations of $(\bar{\tau}, \bar{u})$.

We observe that system (5.1) is not symmetric, so we need to symmetrize it, but instead of the symmetrizer used in §3.3, in this section we use the following

$$S_0(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & \beta(\xi)^{-1} \end{pmatrix}. \tag{5.2}$$

Clearly $S_0(\xi) > 0$ and symmetric, and we define

$$\hat{A}(\xi) := S_0(\xi)A(\xi) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \hat{B}(\xi) := S_0(\xi)B = \begin{pmatrix} 0 & 0 \\ 0 & \beta(\xi)^{-1}\bar{\mu}/\bar{\tau} \end{pmatrix},$$

which are symmetric and $\hat{B} \geq 0$. Then we have the symmetric version of (5.1)

$$S_0(\xi)\hat{U}_t + i\xi\hat{A}(\xi)\hat{U} + \xi^2\hat{B}(\xi)\hat{U} = 0. \tag{5.3}$$

Thus we have the following definition.

Definition 5.1 *Let $S_0, \hat{A}, \hat{B} \in C^\infty(\mathbb{R}; \mathbb{R}^{2\times 2})$ smooth, real matrix functions of $\xi \in \mathbb{R}$. Assume that $S_0, \hat{A}, \hat{B}$ are symmetric, with $S_0 > 0$ and $\hat{B} \geq 0$ for all $\xi \in \mathbb{R}$. A real matrix valued function $K \in C^\infty(\mathbb{R}; \mathbb{R}^{2\times 2})$ is said to be a compensating function for the triplet $(S_0, \hat{A}, \hat{B})$ if*

a) $K(\xi)S_0(\xi)$ is skew-symmetric for all $\xi \in \mathbb{R}$.

b) $\left[K(\xi)\hat{A}(\xi)\right]^s + \hat{B}(\xi) > 0$ for all $\xi \in \mathbb{R}$, $\xi \neq 0$.

Here $[KA]^s = \frac{1}{2}\left(KA + (KA)^T\right)$ is the symmetric part of KA.

Regarding the comment made at the end of §3.4, Humpherys' theory enable us to construct a compensating function for the triplet $(I, \hat{A}, \hat{B})$. One can easily check the following lemma.

Lemma 5.2 *Under the assumptions of definition 5.1. Let us define*

$$\tilde{A}(\xi) := S_0(\xi)^{-1/2}\hat{A}(\xi)S_0(\xi)^{-1/2},$$

$$\tilde{B}(\xi) := S_0(\xi)^{-1/2}\hat{B}(\xi)S_0(\xi)^{-1/2},$$

which are symmetric matrices and $\tilde{B} \geq 0$. Assume that $\tilde{K} = \tilde{K}(\xi)$ is a compensating function for the triplet $(I, \tilde{A}, \tilde{B})$. Then

$$K(\xi) := S_0^{1/2}(\xi)\tilde{K}(\xi)S_0^{-1/2}(\xi), \;\; \xi \in \mathbb{R},$$

is a compensating function for the triplet $\left(S_0, \hat{A}, \hat{B}\right)$.

What we need next is a compensating function for the triplet $(S_0, \hat{A}, \hat{B})$, for S_0, $\hat{A}$, and $\hat{B}$ defined as above. Then by the preceding lemma if we can construct a compensating function for the triplet $(I, \tilde{A}, \tilde{B})$, automatically we obtain one for the triplet $(S_0, \hat{A}, \hat{B})$. Then we use the construction due to Humpherys applied to the triplet $(I, \tilde{A}, \tilde{B})$.
Now, verifying the genuinely coupling condition for $\tilde{A}$ and $\tilde{B}$ it is equivalent to verifying it for A and B. This because by definition

$$\tilde{A}(\xi) = S_0(\xi)^{1/2}A(\xi)S_0(\xi)^{-1/2},$$
$$\tilde{B}(\xi) = S_0(\xi)^{1/2}B(\xi)S_0(\xi)^{-1/2}.$$

Then v in the kernel of $\tilde{B}$ implies $S_0^{-1/2}(\xi)v$ is in the kernel of B. Now if v is also an eigenvector of $\tilde{A}$, we have

$$S_0(\xi)^{1/2}A(\xi)S_0(\xi)^{-1/2}v = \lambda v,$$

which implies

$$A(\xi)S_0(\xi)^{-1/2}v = \lambda S_0(\xi)^{-1/2}v,$$

that is, $S_0(\xi)^{-1/2}v$ is an eigenvector of A. Therefore genuine coupling of A and B implies the one for $\tilde{A}$ and $\tilde{B}$.

One can easily, as we did for $(\hat{A}, \hat{B})$ in §3.4, that A and B satisfy the genuine coupling condition, thus, according to the above remark, such condition also holds for $\tilde{A}$ and $\tilde{B}$. Then we have the following lemma.

Lemma 5.3 *Let us define*

$$K(\xi) := \delta\frac{\xi^2\bar{\mu}}{4\beta(\xi)^2\bar{\tau}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}S_0(\xi)^{-1},$$

where $0 < \delta < 2\bar{k}$. Then:

i) $K(\xi) \in C^\infty\left(\mathbb{R}; \mathbb{R}^{2\times 2}\right)$.

ii) $K(\xi)$ *is a compensating matrix function for the triplet* $(S_0, \hat{A}, \hat{B})$.

iii) $K(\xi)$ *is uniformly bounded in* ξ*, that is, there exists* $C > 0$ *(independent of* ξ*) such that*

$$|K(\xi)| \leq C, \quad \forall \xi.$$

Proof. Clearly, because of the form of $\beta(\xi)$ and S_0, the first assertion is true.
Next we proceed to prove ii). By the definition of $K(\xi)$, we have

$$K(\xi)S_0(\xi) = \delta\frac{\xi^2\bar{\mu}}{4\beta(\xi)^2\bar{\tau}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

which is a skew-symmetric matrix. For the product $K(\xi)\hat{A}(\xi)$, we get

$$\begin{aligned} K(\xi)\hat{A}(\xi) &= \delta\frac{\xi^2\bar{\mu}}{4\beta(\xi)^2\bar{\tau}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} S_0(\xi)^{-1}S_0(\xi)A(\xi)\\ &= \delta\frac{\xi^2\bar{\mu}}{4\beta(\xi)^2\bar{\tau}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} A(\xi)\\ &= \delta\frac{\xi^2\bar{\mu}}{4\beta(\xi)^2\bar{\tau}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ -\beta(\xi) & 0\end{pmatrix}\\ &= \delta\frac{\xi^2\bar{\mu}}{4\beta(\xi)^2\bar{\tau}}\begin{pmatrix}\beta(\xi) & 0\\ 0 & -1\end{pmatrix}, \end{aligned}$$

which is symmetric, then $[K\hat{A}]^s = K\hat{A}$ and

$$\begin{aligned} \Big[K(\xi),\hat{A}(\xi)\Big]^s + \hat{B}(\xi) &= K(\xi)\hat{A}(\xi) + B(\xi)\\ &= \delta\frac{\xi^2\bar{\mu}}{4\beta(\xi)^2\bar{\tau}}\begin{pmatrix}\beta(\xi) & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \beta(\xi)^{-1}\bar{\mu}/\bar{\tau}\end{pmatrix}\\ &= \frac{\xi^2\bar{\mu}}{4\beta(\xi)^2\bar{\tau}}\left[\delta\begin{pmatrix}\beta(\xi) & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 4\beta(\xi)/\xi^2\end{pmatrix}\right]\\ &= \frac{\xi^2\bar{\mu}}{4\beta(\xi)^2\bar{\tau}}\begin{pmatrix}\delta\beta(\xi) & 0\\ 0 & (4\beta(\xi)/\xi^2)-\delta\end{pmatrix}, \end{aligned}$$

so that $[K,\hat{A}] + \hat{B}(\xi) > 0$ if $(4\beta(\xi)/\xi^2) - \delta > 0$, which holds when $0 < \delta < 2\bar{k}$, since we have that

$$\frac{4\beta(\xi)}{\xi^2} - \delta = \frac{4\bar{q} + \xi^2\bar{k}}{\xi^2} - \delta \geq 4\bar{k} - \delta.$$

Thus the proof of ii) is complete.

To show iii), let us observe that

$$S_0(\xi)^{-1} = \begin{pmatrix}1 & 0\\ 0 & \beta(\xi)\end{pmatrix}.$$

Then we have

$$|K(\xi)| = \delta\frac{\xi^2\bar{\mu}}{4\beta(\xi)^2\bar{\tau}}O(\beta(\xi)) = \tilde{C}\cdot O\left(\frac{\xi^2}{\beta(\xi)}\right) = \tilde{C}\cdot O(1) = C,$$

where $C > 0$ is independent of ξ, and we have used that $\beta(\xi) = \bar{q} + \xi^2\bar{k}$. Thus the proof of the lemma is complete. ■

Next, we state and prove the results regarding the decay of the solution to the linearized system. The statements and proofs are the same as in [3] , however, for the sake of completeness we include the proofs.

Let us $\langle , \rangle$ denote the inner product in $\mathbb{C}^n$, and U the solution to the linearization of (3.10). Then we have the following lemma.

Lemma 5.4 *There exists $k > 0$ such that the solution to the linear system (3.12) satisfies*

$$|\hat{U}(\xi,t)| \leq C|\hat{U}(\xi,0)|\exp\left(-\frac{k\xi^2 t}{1+\xi^2}\right), \tag{5.4}$$

for all $t \geq 0$, $\xi \in \mathbb{R}$ and some uniform constant $C > 0$.

Proof. We apply the Fourier transform to (3.12) and obtain (5.1); then we multiply by S_0 to obtain the symmetric system (5.3). Using that the coefficients matrices are symmetric, if we take the inner product of (5.3) with $\hat{U}$ and then take the real part, we obtain

$$\frac{1}{2}\partial_t\langle \hat{U}, S_0\hat{U}\rangle + \xi^2\langle \hat{U}, \hat{B}\hat{U}\rangle = 0. \tag{5.5}$$

Now multiply (5.3) by $-i\xi K$ and take the inner product with $\hat{U}$. We obtain

$$-\langle \hat{U}, i\xi K S_0 \hat{U}_t\rangle + \xi^2\langle \hat{U}, K\hat{A}\hat{U}\rangle - \langle \hat{U}, i\xi^3 K\hat{B}\hat{U}\rangle = 0. \tag{5.6}$$

Using the fact that KS_0 is skew-symmetric one can easily verify that

$$\mathrm{Re}\langle \hat{U}, i\xi K S_0\hat{U}_t\rangle = \frac{1}{2}\xi\partial_t\langle \hat{U}, iKS_0\hat{U}\rangle.$$

Also, we have the relation

$$\begin{aligned}
\mathrm{Re}\left(\langle \hat{U}, K\hat{A}\hat{U}\rangle\right) &= \frac{1}{2}\left(\langle \hat{U}, K\hat{A}\hat{U}\rangle + \overline{\langle \hat{U}, K\hat{A}\hat{U}\rangle}\right)\\
&= \frac{1}{2}\left(\langle \hat{U}, K\hat{A}\hat{U}\rangle + \langle K\hat{A}\hat{U}, \hat{U}\rangle\right)\\
&= \frac{1}{2}\left(\langle \hat{U}, K\hat{A}\hat{U}\rangle + \langle \hat{U}, (K\hat{A})^T\hat{U}\rangle\right)\\
&= \langle \hat{U}, [K\hat{A}]^s\hat{U}\rangle.
\end{aligned}$$

Thus, we take the real part in (5.6) and get

$$-\frac{1}{2}\xi\partial_t\langle \hat{U}, iKS_0\hat{U}\rangle + \xi^2\langle \hat{U}, [K\hat{A}]^s\hat{U}\rangle = \mathrm{Re}(\langle \hat{U}, i\xi^3 K\hat{B}\hat{U}\rangle),$$

Since $\hat{B} \geq 0$, we obtain, using (4.31), the estimate

$$-\frac{1}{2}\xi\partial_t\langle \hat{U}, iKS_0\hat{U}\rangle + \xi^2\langle \hat{U}, [K\hat{A}]^s\hat{U}\rangle \leq \epsilon\xi^2|\hat{U}|^2 + C_\epsilon\xi^4\langle \hat{U}, \hat{B}\hat{U}\rangle, \tag{5.7}$$

for any $\epsilon > 0$ and where $C_\epsilon > 0$ is a uniform constant depending only on $\epsilon > 0$ and $|K\hat{B}^{1/2}|$. Before we proceed, we note that $|K\hat{B}^{1/2}|$ is uniformly bounded in ξ since

$$\begin{aligned}
K(\xi)\hat{B}^{1/2}(\xi) &= \frac{\delta\xi^2\bar{\mu}}{4\beta^2\bar{\tau}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & \sqrt{\frac{\bar{\mu}}{\bar{\tau}}}\beta^{-1/2}\end{pmatrix}\\
&= \frac{\delta\xi^2\bar{\mu}}{4\beta^2\bar{\tau}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & \sqrt{\frac{\bar{\mu}}{\bar{\tau}}}\beta^{1/2}\end{pmatrix},
\end{aligned}$$

so that

$$|K\hat{B}^{1/2}| = \frac{\xi^2}{\beta^2}\mathrm{O}(\beta^{1/2}) = \mathrm{O}\left(\frac{\xi^2}{\beta^{3/2}}\right) = \mathrm{O}\left(\frac{\xi^2}{\xi^3}\right) = \mathrm{O}\left(\frac{1}{\xi}\right).$$

Now multiply equation (5.5) by $1+\xi^2$, equation (5.7) by $\gamma > 0$ and add them up. The result is

$$\begin{aligned}
&\frac{1}{2}\partial_t\left((1+\xi^2)\langle \hat{U}, S_0\hat{U}\rangle - \gamma\xi\langle \hat{U}, iKS_0\hat{U}\rangle\right) + \xi^4\langle \hat{U}, \hat{B}\hat{U}\rangle\\
&\quad + \xi^2\left(\gamma\langle \hat{U}, [K\hat{A}]^s\hat{U}\rangle + \langle \hat{U}, \hat{B}\hat{U}\rangle\right)\\
&\quad \leq \epsilon\gamma\xi^2|\hat{U}|^2 + \gamma C_\epsilon\xi^4\langle \hat{U}, \hat{B}\hat{U}\rangle
\end{aligned} \tag{5.8}$$

Now define

$$R := \langle \hat{U}, S_0 \hat{U} \rangle - \frac{\gamma \xi}{1+\xi^2} \langle \hat{U}, iK\hat{A}\hat{U} \rangle.$$

Since S_0 is symmetric and KS_0 is skew-symmetric, R is real. Also, $S_0 > 0$, so that we can find $C_0 > 0$ such that $\langle \hat{U}, S_0 \hat{U} \rangle \geq C_0 |\hat{U}|^2$. Thus, we can easily find $\gamma_0 > 0$, sufficiently small, such that if $0 < \gamma < \gamma_0$, then

$$\frac{1}{C_1} |\hat{U}|^2 \leq R \leq C_1 |\hat{U}|^2,$$

for some uniform $C_1 > 0$.

Now, from the property b) of definition (5.1) for the compensating function K, there exists $\tilde{C}_0 > 0$ such that $\langle \hat{U}, ([K\hat{A}]^s + \hat{B})\hat{U} \rangle \geq \tilde{C}_0 |\hat{U}|^2$. Then if we take $0 < \gamma < 1$ we arrive at

$$\langle \hat{U}, (\gamma [K\hat{A}]^s + \hat{B})\hat{U} \rangle \geq \tilde{C}_0 \gamma |\hat{U}|^2.$$

Now, we choose $\epsilon = \tilde{C}_0/2$ and $0 < \gamma < \min\{1, \gamma_0, 1/C_\epsilon\}$. Thus from the last inequality and (5.8) we obtain

$$\frac{1}{2} \partial_t R + \frac{1}{2} \left(\frac{\xi^2}{1+\xi^2} \right) \gamma \tilde{C}_0 |\hat{U}|^2 + \frac{(1 - \gamma C_\epsilon)}{1+\xi^2} \xi^4 \langle \hat{U}, \hat{B}\hat{U} \rangle \leq 0.$$

Using the lower bound for R and that $\langle \hat{U}, \hat{B}\hat{U} \rangle \geq 0$, the above expression implies

$$\partial_t R + \frac{k\xi^2}{1+\xi^2} R \leq 0,$$

where $k = \tilde{C}_0 \gamma / C_1$. This inequality together with the Gronwall theorem imply the desire estimate (5.4). ■

Theorem 5.5 *Suppose that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, with $s \geq 2$. Then the solution to the Cauchy problem for the linear system (3.12) with $U(x,0) = U_0$ satisfies the decay rate*

$$\|\partial_x^l U\|_{L^2}^2 \leq C \left(e^{-kt} \|\partial_x^l U_0\|_{L^2}^2 + (1+t)^{-(l+1/2)} \|U_0\|_{L^1}^2 \right), \tag{5.9}$$

for $0 \leq l \leq s-1$ and some uniform $C > 0$.

Proof. Multiplying estimate (5.4) by ξ^{2l} and then integrating yields

$$\int_{\mathbb{R}^2} \xi^{2l} |\hat{U}(\xi,t)|^2 d\xi \leq C \int_{\mathbb{R}} \xi^{2l} |\hat{U}(\xi,0)|^2 \exp\left(-\frac{2k\xi^2 t}{1+\xi^2} \right) d\xi =: C\left(R_1(t) + R_2(t)\right),$$

where R_1 and R_2 denote the integral on the right side computed on the sets $\xi \in (-1,1)$ and $|\xi| > 1$, respectively. Since for $\xi \in (-1,1)$ it holds $\frac{1}{2}\xi^2 \leq \xi^2/(1+\xi^2)$, we have the estimate for $R_1(t)$

$$\begin{aligned} R_1(t) = \int_{-1}^{1} \xi^{2l} |\hat{U}(\xi,0)|^2 \exp\left(-\frac{2k\xi^2 t}{1+\xi^2} \right) d\xi &\leq \sup_{\xi \in \mathbb{R}} |\hat{U}_0(\xi)|^2 \int_{-1}^{1} \xi^{2l} e^{-k\xi^2 t} d\xi \\ &\leq \|U_0\|_{L^1}^2 \int_{-1}^{1} \xi^{2l} e^{-k\xi^2 t} d\xi. \end{aligned}$$

Now, we claim that

$$A(t) := (1+t)^{l+1/2} \int_{-1}^{1} \xi^{2l} e^{-k\xi^2 t} d\xi \tag{5.10}$$

is continuous and uniformly bounded for all $t > 0$. Then we obtain that

$$R_1(t) \leq C\,(1+t)^{-(l+1/2)}\, \|U_0\|_{L^1}^2,$$

for some $C > 0$ and all $t \geq 0$. Now, for $\xi^2 \geq 1$, $2\xi^2 \geq 1+\xi^2$, then it holds $\exp(-2k\xi^2 t/(1+\xi^2)) \leq e^{-kt}$. Therefore together with the Pancherel's theorem, we obtain

$$\begin{aligned} R_2(t) &= \int_{|\xi|\geq 1} \xi^{2l} |\hat{U}(\xi,0)|^2 \exp\left(-\frac{2k\xi^2 t}{1+\xi^2}\right) d\xi \\ &\leq e^{-kt} \int_{\mathbb{R}} \xi^{2l} |\hat{U}_0(\xi)|^2 d\xi \\ &\leq e^{-kt} \|\partial_x^l U_0\|_{L^2}^2. \end{aligned}$$

Combining the estimates for $R_1(t)$ and $R_2(t)$ we get (5.9). ■

Before stating a corollary of theorem (5.5), we prove the claim made in the proof above. Clearly $A(t) \geq 0$ and continuous for all $t \geq 0$, then for $R > 0$, we can find $C_R(R)$ such that

$$A(t) \leq C_R, \ \forall t \in [0, R].$$

Now, making the change of variable $y = \xi^2 t$, one gets

$$\begin{aligned} A(t) &= 2\,(1+t)^{l+1/2} \int_0^1 \xi^{2l} e^{-k\xi^2 t}\, d\xi \\ &= 2\,(1+t)^{l+1/2} \int_0^t \left(\frac{y}{t}\right)^l \frac{e^{-ky}}{2t^{1/2} y^{1/2}}\, dy, \quad dy = 2\xi t d\xi = 2\left(\frac{y}{t}\right)^{1/2} t = 2y^{1/2} t^{1/2} \\ &= \frac{(1+t)^{l+1/2}}{t^{l+1/2}} \int_0^t y^{l-1/2} e^{-ky} dy \\ &= \left(1+\frac{1}{t}\right)^{l+1/2} \int_0^t y^{l-1/2} e^{-ky} dy. \end{aligned}$$

Since we can bound the term $(1+1/t)^{l+1/2}$ for t large, it is enough to show that the limit

$$\lim_{t\to\infty} \int_0^t y^{l-1/2} e^{-ky} dy$$

exists. To see that, let us remember that for properties of the exponential function, for R sufficiently large we have

$$y^{l-1/2} \leq C e^{ky/2}, \text{ for } \quad y \geq R,$$

and $C > 0$ a constant. Therefore

$$\begin{aligned} A(t) &= \left(1+\frac{1}{t}\right)^{l+1/2} \int_0^t y^{l-1/2} e^{-ky} dy \\ &= \left(1+\frac{1}{t}\right)^{l+1/2} \left(\int_0^R y^{l-1/2} e^{-ky} dy + \int_R^t y^{l-1/2} e^{-ky} dy\right) \\ &\leq \left(1+\frac{1}{t}\right)^{l+1/2} \left(\int_0^R y^{l-1/2} e^{-ky} dy + C\int_R^t e^{-ky/2} dy\right) \\ &\leq \tilde{C}_R + \left(1+\frac{1}{t}\right)^{l+1/2} \left(-C\frac{2}{k} e^{-ky/2}\right)\Bigg|_R^t = \tilde{C}_R + \left(1+\frac{1}{t}\right)^{l+1/2} \frac{2C}{k}\left(e^{-\frac{k}{2}R} - e^{-\frac{k}{2}t}\right) \\ &\leq \tilde{C}_R + \left(1+\frac{1}{t}\right)^{l+1/2} \frac{2C}{k} e^{-\frac{k}{2}R} = C, \text{ for all } t \in [R, \infty). \end{aligned}$$

which completes the proof of the claim.

Corollary 5.6 *Let $\bar{U} = (\bar{\tau}, \bar{u})$ be a constant equilibrium state. If $U_0 - \bar{U} \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, with $s \geq 2$, is a initial perturbation (with finite energy ang mass) of the equilibrium state $\bar{U}$ then the solution $U - \bar{U}$ to the linearized equation around $\bar{U}$ satisfy the decay estimate*

$$\|\partial_x^l (U - \bar{U})\|_{L^2}^2 \leq C \left(e^{-kt} \|\partial_x^l (U_0 - \bar{U})\|_{L^2}^2 + (1+t)^{-(l+1/2)} \|U_0 - \bar{U}\|_{L^1}^2 \right). \tag{5.11}$$

Proof. Let $\bar{U} = (\bar{\tau}, \bar{u})$ be a constant equilibrium state, and suppose $U = \bar{U} + V$ is a solution of (3.10). Such system, we can write it in the quasilinear form (3.11):

$$V_t + D_1(\bar{U})V_x + D_2(\bar{U})V_{xx} + D_3(\bar{U})V_{xxx} = \mathcal{N},$$

where $\mathcal{N}$ contains the non-linear terms and $V = U - \bar{U}$ represents a perturbation of the equilibrium state. If we discard the non-linear terms we arrive at the constant coefficient, linearized system (3.12) for the perturbation V, that is

$$V_t + V_1(\bar{U})V_x + D_2(\bar{U})V_{xx} + D_3(\bar{U})V_{xxx} = 0.$$

Thus the hypotheses of Theorem (5.5) are satisfied, and for any solution $V = U - \bar{U}$ of the linearized system (3.12) with initial condition $U_0 - \bar{U} \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ for some $s \geq 2$ the desired linear decay holds. ■

6 Conclusions

For the one-dimensional isothermal compressible model for fluids of Korteweg type we have done a study of its dissipative structure in the sense of Humpherys. Such study has not been reported in the literature, and this is one of the contributions of the present work.

It is worth to note the detailed analysis made of the well-posedness of the model. Although such analysis is based on the one made by Hattori and Li, the presented here has been adapted to the one-dimensional case and improved in some senses. For example we gave detailed proofs for the a priori energy estimates and presented a more direct fixed point argument for the local existence of the nonlinear problem.

We have obtained linear decay of the solutions for the linearized model around constants states, which is the first step in obtaining a nonlinear result. In that direction we are working to publish a paper.

References

[1] E. C. Aifantis and J. B. Serrin. The mechanical theory of fluid interfaces and Maxwell's rule. *J. Colloid Interface Sci.*, 96(2):517–529, 1983.

[2] D. M. Anderson, G. B. McFadden, and A. A. Wheeler. Diffuse-interface methods in fluid mechanics. *Annu. Rev. Fluid Mech.*, 30(1):139–165, 1998.

[3] F. Angeles, C. Málaga, and R. G. Plaza. Strict dissipativity of Cattaneo–Christov systems for compressible fluid flow. *J. Phys. A: Math. Theor.*, 53(6):065701, 2020.

[4] S. Benzoni-Gavage, R. Danchin, and S. Descombes. Well-posedness of one-dimensional Korteweg models. *Electron. J. Differ. Equ.*, 2006:35 pp, 2006.

[5] J. E. Dunn and J. Serrin. On the thermomechanics of interstitial working. *Arch. Ration. Mech. Anal.*, 88(2):95–133, 1985.

[6] L. C. Evans. *Partial differential equations.* American Mathematical Society, Providence, R.I., 2010.

[7] K. O. Friedrichs. Symmetric hyperbolic linear differential equations. *Commun. Pure Appl. Math.*, 7(2):345–392, 1954.

[8] H. Hattori and D. Li. Solutions for two-dimensional system for materials of Korteweg type. *SIAM J. Math. Anal.*, 25(1):85–98, 1994.

[9] H. Hattori and D. Li. Global solutions of a high dimensional system for Korteweg materials. *J. Math. Anal. Appl.*, 198(1):84–97, 1996.

[10] J. Humpherys. Admissibility of viscous-dispersive systems. *J. Hyperbolic Differ. Equ.*, 2(04):963–974, 2005.

[11] J. L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications*, volume 1. Springer-Verlag, New York, 1972.

[12] J. C. Robinson. *Infinite-dimensional dynamical systems: an introduction to dissipative parabolic PDEs and the theory of global attractors*, volume 28. Cambridge University Press, 2001.

[13] Y. Shizuta and S. Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14(2):249–275, 1985.

[14] M. Slemrod. Admissibility criteria for propagating phase boundaries in a van der Waals fluid. *Arch. Ration. Mech. Anal.*, 81(4):301–315, 1983.

[15] D. H. Wagner. Equivalence of the Euler and Lagrangian equations of gas dynamics for weak solutions. *J. Differ. Equ.*, 68(1):118–136, 1987.