Métodos numéricos en teoría KAM por el método de la parametrización

Renato Calleja y Pedro Porras

2005 de la Llave, Jorba, Villanueva, Gonzalez

2016 Haro, Canadell, Figueras, Luque, Mondelo

http://www.maia.ub.es/dsg/param/

Ecuaciones cohomológicas

$$\mathcal{L}\varphi(\theta)=\varphi(\theta)-\varphi(\theta+\omega)=\eta(\theta)$$

$$\eta:\mathbb{T}^n\longrightarrow\mathbb{R}\,,\quad \int_{\mathbb{T}^n}\eta(\theta)\,d\theta=0$$

$$\eta(\theta)=\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\hat{\eta}_k\, e^{2\pi i k\cdot\theta}$$

$$\varphi(\theta)=\sum\cdots$$

En Fourier

$$\hat{\varphi}_k-\hat{\varphi}_k e^{2\pi i k\omega}=\hat{\eta}_k$$

$$\hat{\varphi}_k\left(1-e^{2\pi i k\cdot\omega}\right)=\hat{\eta}_k$$

$$k\neq 0\qquad \hat{\varphi}_k=\frac{\hat{\eta}_k}{1-e^{2\pi i k\cdot\omega}}$$

$$\varphi(\theta)=\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\frac{\hat{\eta}_k}{1-e^{2\pi i k\cdot\omega}}\,e^{2\pi i k\cdot\theta}$$

Persistencia de toros para mapeos simplécticos

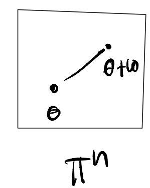
$$K:\mathbb{T}^n\longrightarrow\mathbb{T}^n\times\mathbb{R}^n$$

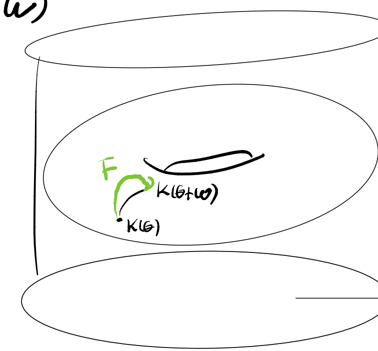
$$\mathcal{M}=\mathbb{T}^n\times\mathbb{R}^n\,,\quad\Omega$$

$$F:\mathcal{M}\longrightarrow\mathcal{M}\,,\quad F^*\Omega=\Omega$$

Ecuación de invarianza para ω fijo

$$F(K(\theta))=K(\theta+\omega)$$





$K = K(\mathbb{T}^n)$ es Lagrangiano

$$K^*\Omega = 0$$

Empezamos de una soln aproximada

$$K_0 : \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{R}^n$$

$$F \circ K_0(\theta) - K_0(\theta+\omega) = E_0(\theta)$$

$$K_1(\theta) = K_0(\theta) + \Delta(\theta)$$

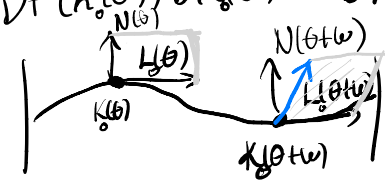
$$\begin{aligned} E_1(\theta) &= F \circ K_1(\theta) - K_1(\theta+\omega) \\ &= F(K_0(\theta)) + DF(K_0(\theta))\Delta(\theta) + O(\|\Delta\|^2) \\ &\quad - K_0(\theta+\omega) - \Delta(\theta+\omega) \\ &= E_0(\theta) + DF(K_0(\theta))\Delta(\theta) - \Delta(\theta+\omega) + O(\|\Delta\|^2) \end{aligned}$$

$$DF(K_0(\theta))\Delta(\theta) - \Delta(\theta+\omega) = -E_0(\theta)$$

Un marco de referencia

$$F(K_0(\theta)) = K_0(\theta+\omega) + E_0$$

$$DF(K_0(\theta))\, DK_0(\theta) = DK_0(\theta+\omega) + O(\|E_0\|)$$



$$L_0(\theta) = DK_0(\theta)$$

Queremos construir un marco simpléctico que nos ayude.

$$P(\theta) = (L(\theta) \vdots N(\theta))$$

Quiero $P(\theta)^T J(K(\theta)) P(\theta) = J_0$

$$N(\theta) = J(K(\theta)) L(\theta) G(\theta)^{-1}$$

$$G(\theta) = L(\theta)^T L(\theta)$$

$$DF(K_0(\theta))\, P_0(\theta) = P_0(\theta+\omega) \begin{pmatrix} I & T_0(\theta) \\ 0 & I \end{pmatrix} + O(\|E_0\|)$$

Cambio coordenadas

$$\Delta(\theta) = P_0(\theta)\xi(\theta) \qquad \Delta = \begin{pmatrix} \Delta^x \\ \Delta^y \end{pmatrix}$$

$$\xi = \begin{pmatrix} \xi^N \\ \xi^L \end{pmatrix}$$

$$DF(K_0(\theta))P_0(\theta)\xi(\theta) - P_0(\theta+\omega)\xi(\theta+\omega) = -E_0(\theta)$$

$$P_0(\theta+\omega)\begin{pmatrix} I & T(\theta) \\ 0 & I \end{pmatrix}\xi(\theta) - P_0(\theta+\omega)\xi(\theta+\omega) = -E_0(\theta)$$

$$P^{-1}(\theta) = -J_0 P(\theta)^T J(K(\theta))$$

$$\eta(\theta) = J_0 P(\theta+\omega)^T J(K(\theta+\omega))E_0(\theta)$$

$$= \begin{pmatrix} \eta^L(\theta) \\ \eta^N(\theta) \end{pmatrix}$$

$$\begin{pmatrix} I & T(\theta) \\ 0 & I \end{pmatrix}\begin{pmatrix} \xi^L(\theta) \\ \xi^N(\theta) \end{pmatrix} - \begin{pmatrix} \xi^L(\theta+\omega) \\ \xi^N(\theta+\omega) \end{pmatrix} = \begin{pmatrix} \eta^L(\theta) \\ \eta^N(\theta) - \langle\eta^N\rangle \end{pmatrix}$$

Hay 2 comps

$$\xi^N(\theta) - \xi^N(\theta+\omega) = \eta^N(\theta) - \langle\eta^N\rangle$$

$\langle\xi^N\rangle = C_N$ grado de libertad!

$$\xi^L(\theta) - \xi^L(\theta+\omega) = \eta^L(\theta) - T(\theta)\xi^N(\theta)$$

siempre que $\det\langle T\rangle \neq 0$

Escogemos C_N para que

$$\langle\eta^L - T\xi^N\rangle = 0$$

La nueva

$$K_1(\theta) = K_0(\theta) + P_0(\theta)\begin{pmatrix} \xi^L \\ \xi^N \end{pmatrix}$$

$$E_1 = F\circ K_1 - K_1\circ R_\omega$$

$$R_\omega = \theta+\omega$$

$$\mathcal{O}(E_1) = \mathcal{O}(E_0^2)$$

Cómo implementar

f en $\mathbb{T}$, $\theta_j = \frac{j}{N}$ con $0 \leq j \leq N$

$$\{\hat{f}_k\} = DFT(\{f_j\}) \qquad \hat{f}_k = \frac{1}{N} \sum_j f_j e^{-2\pi i k \theta_j}$$

Aquí DFT produce un polinomio trigonométrico

$$f(\theta_j) = \sum_k \hat{f}_k e^{2\pi i k \cdot \theta_j}$$

$$f(\theta) \simeq \sum_{k'} \hat{f}_{k'} e^{2\pi i k' \cdot \theta_j}$$

$$k' = \begin{cases} k & 0 \leq k \leq N \\ k - N & \text{si } N/2 \leq k < N \end{cases}$$

$$\{\widehat{(D_\theta f)}_k\} = \{2\pi i k' \hat{f}_k\}$$

$$R_\omega(\theta) = \theta + \omega$$

$$\{\widehat{(f \circ R_\omega)}\} = \{e^{2\pi i k' \cdot \omega} \hat{f}_k\}$$

$$\varphi(\theta) - \varphi(\theta+\omega) = \eta(\theta)$$

$$\hat{\varphi}_k = \begin{cases} (1 - e^{2\pi i k' \omega})^{-1} \hat{\eta}_k & \text{si } k \neq 0 \\ 0 & \text{si } k = 0 \end{cases}$$

Puedo resolver cohomológicas y tomar derivadas.

$$DF(K_0(\theta))\Delta(\theta) - \Delta(\theta+\omega) = -E_0(\theta)$$

Puedo invertir la matriz en $O(N^3)$

$$\begin{pmatrix} I & T(\theta) \\ 0 & I \end{pmatrix}\begin{pmatrix}\xi^L(\theta)\\ \xi^N(\theta)\end{pmatrix} - \begin{pmatrix}\xi^L(\theta+\omega)\\ \xi^N(\theta+\omega)\end{pmatrix} = \begin{pmatrix}\eta^L(\theta)\\ \eta^N(\theta)-\langle\eta^N\rangle\end{pmatrix}$$

$$\left.\begin{array}{ll}\text{En espacio real} & O(N)\\ \text{En fourier} & O(N) \\ \text{FFT} & O(N\log N)\end{array}\right\} O(N\log N)$$